MthT 430 Projects Chapter 7a – Three Hard Theorems

The point of the projects is to show that all the Hypotheses in the Three Hard Theorems are necessary to assure the Conclusion. Construct (by pictures or formulas) functions which violate one of the hypotheses and do not satisfy the conclusion.

(CFIVP) Continuous Functions on Intervals Have the Intermediate Value Property

Theorem 1. *If f is continuous on $[a, b]$ and $f(a) < 0 < f(b)$, then there is some x in $[a, b]$ such that $f(x) = 0$.*

1. Construct a function f on $[0, 1]$ such that

- f is continuous on $[0, 1]$ except at $x = \frac{1}{2}$.
- $f(0) < 0 < f(1)$.
- There is no $x \in [0, 1]$ such that $f(x) = 0$.

(CFCIB) Continuous Functions on Closed Intervals are Bounded

Theorem 2. *If f is continuous on $[a, b]$, then f is bounded above on $[a, b]$, that is, there is some number N such that $f(x) \leq N$ for all x in $[a, b]$.*

2. Construct a function f on $[0, 1]$ such that

- f is continuous on $[0, 1]$ except at $x = \frac{1}{2}$.
- $f(0) < 0 < f(1)$.
- f is not bounded on $[0, 1]$.

(CFCIMAX) Continuous Functions on Closed Intervals assume a Maximum Value for the Interval

Theorem 3. *If f is continuous on $[a, b]$, then there is a number y in $[a, b]$ such that $f(y) \geq f(x)$ for all x in $[a, b]$*

3. Construct a function f on $[0, 1]$ such that
   - f is continuous on $[0, 1]$ except at $x = \frac{1}{2}$.
   - $f(0) < 0 < f(1)$.
   - There is no number y in $[0, 1]$ such that $f(y) \geq f(x)$ for all x in $[0, 1]$.
   - There is no number y in $[0, 1]$ such that $f(y) \leq f(x)$ for all x in $[0, 1]$.
4. Construct a function f on $[0, 1]$ such that
   - f is continuous on $[0, 1]$ except at $x = \frac{1}{2}$.
   - $f(0) < 0 < f(1)$.
   - There is no number y in $[0, 1]$ such that $f(y) \geq f(x)$ for all x in $[0, 1]$.
   - There is no number y in $[0, 1]$ such that $f(y) \leq f(x)$ for all x in $[0, 1]$.
5. Construct a function f on $[0, 1)$ such that
   - f is continuous on $[0, 1)$.
   - f is bounded on $[0, 1)$.
   - There is no number y in $[0, 1)$ such that $f(y) \geq f(x)$ for all x in $[0, 1)$.
   - There is no number y in $[0, 1)$ such that $f(y) \leq f(x)$ for all x in $[0, 1)$.