MthT 430 Notes Chap 7b Hard Theorems – Proofs by Binary Expansion

For this discussion, we shall assume:

(P13–BIN) Binary Expansions Converge. *Every binary expansion represents a real number x: every infinite series of the form*

$$c_1 2^{-1} + c_2 2^{-2} + \ldots + c_k 2^{-k} + \ldots, \quad c_k \in \{0,1\},$$

converges to a real number x in $[0,1]$ and write the binary expansion of x as

$$x = ._{\text{bin}} c_1 c_2 \ldots.$$

Continuous Functions on Intervals Have the Intermediate Value Property

Theorem 1. *If f is continuous on $[a,b]$ and $f(a) < 0 < f(b)$, then there is some x in $[a,b]$ such that $f(x) = 0$.*

In numerical analysis, the following is known as finding the root of $f(x) = 0$ by the *method of bisection*.

We will show this result by constructing the binary expansion of a number $x \in (a,b)$ such that $f(x) = 0$.

Without loss of generality, $[a,b] = [0,1]$. The rough idea is to ask: If there is such an x, is x in the left half or in the right half of $[0,1]$, and then proceed by recursion (induction).

Let $m_1 = \frac{1}{2}$ be the midpoint of $[0,1]$. Ask the question: Is $f(m_1) < 0$, $= 0$, or > 0.

Cases:

- If $f(m_1) = 0$, let $x = m_1 = 0._{\text{bin}}1$. STOP! $f(x) = 0$ as desired.
- If $f(m_1) < 0$, the function changes sign on $(m_1, 1)$ so we look for the root on $(m_1, 1)$. Let

$$\begin{aligned} a_1 &= m_1, \\ b_1 &= 1, \\ c_1 &= 1, \\ s_1 &= 0._{\text{bin}} c_1 \\ &= 0._{\text{bin}} 1 \\ &= a_1. \end{aligned}$$

  Note that

$$b_1 - a_1 = \frac{1}{2^1},$$
$$f(a_1) < 0 < f(b_1).$$

- If $f(m_1) > 0$, the function changes sign on $(0, m_1)$ so we look for the root on $(0, m_1)$. Let

$$\begin{aligned} a_1 &= 0, \\ b_1 &== m_1 \\ &= \frac{1}{2}, \\ &= a_1 + \frac{1}{2} \\ c_1 &= 0, \\ s_1 &= 0._{\text{bin}} c_1 \\ &= 0._{\text{bin}} 0 \\ &= a_1. \end{aligned}$$

Note that

$$\begin{aligned} b_1 - a_1 &= \frac{1}{2^1}, \\ f(a_1) < 0 &< f(b_1). \end{aligned}$$

We think of c_1 as the first binary digit in the expansion of x.

Suppose that $a_n, b_n = a_n + \frac{1}{2^n}$, $s_n = a_n = 0._{\text{bin}} c_1 \ldots c_n$ have been constructed so that

$$f(a_1) < 0 < f(b_1),$$

, let $m_n = a_n + \frac{1}{2^{n+1}} = \frac{1}{2}(a_n + b_n)$. Ask the question: Is $f(m_n) < 0$, $= 0$, or > 0.

Cases:

- If $f(m_n) = 0$, let $x = m_n = s_n + \frac{1}{2^{n+1}} = 0._{\text{bin}} c_1 \ldots c_n 1$. STOP! $f(x) = 0$ as desired.

- If $f(m_n) < 0$, the function changes sign on (m_n, b_n) so we look for the root on (m_n, b_n). Let

$$\begin{aligned} a_{n+1} &= m_n, \\ b_{n+1} &= b_n, \\ c_{n+1} &= 1, \\ s_1 &= 0._{\text{bin}} c_1 \ldots c_n c_{n+1} \\ &= 0._{\text{bin}} c_1 \ldots c_n 1 \\ &= a_{n+1}. \end{aligned}$$

Note that

$$\begin{aligned} b_{n+1} - a_{n+1} &= \frac{1}{2^{n+1}}, \\ f(a_{n+1}) < 0 &< f(b_{n+1}). \end{aligned}$$

- If $f(m_n) > 0$, the function changes sign on (a_n, m_n) so we look for the root on (a_n, m_n). Let

$$\begin{aligned} a_{n+1} &= a_n, \\ b_{n+1} &= m_n, \\ c_{n+1} &= 0, \\ s_1 &= 0._{\text{bin}} c_1 \ldots c_n c_{n+1} \\ &= 0._{\text{bin}} c_1 \ldots c_n 0 \\ &= a_{n+1}. \end{aligned}$$

  Note that

$$\begin{aligned} b_{n+1} - a_{n+1} &= \frac{1}{2^{n+1}}, \\ f(a_{n+1}) < 0 &< f(b_{n+1}). \end{aligned}$$

If the process does not stop, we have that, for all n,

$$f(s_n) < 0 < f\left(s_n + \frac{1}{2^{n+1}}\right).$$

Let

$$\begin{aligned} x &= \lim_{n \to \infty} s_n \\ &= 0._{\text{bin}} c_1 c_2 \ldots c_n \ldots. \end{aligned}$$

We have that $f(x) = 0$ since

$$\begin{aligned} f(x) &= \lim_{n \to \infty} f(s_n) \\ &\leq 0, \\ f(x) &= \lim_{n \to \infty} f\left(s_n + \frac{1}{2^{n+1}}\right) \\ &\geq 0. \end{aligned}$$

The Bolzano–Weierstraß Theorem

Theorem (Bolzano–Weierstraß). *Let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence of points in.* $[0, 1]$. *Then there is an* x *in* $[0, 1]$ *which is a limit point*[3] *of the sequence* $\{x_n\}_{n=1}^{\infty}$.

The proof will construct a binary expansion for x.

[3] A point x is a limit point of the sequence if for every $\epsilon > 0$, infinitely many terms of the sequence are within ϵ of x. Alternately, there is a subsequence which converges to x. A more informal idea is to say that infinitely many terms are as close as desired to x.

Now – either infinitely many terms of the sequence are 1, in which case $x = 1 = 0._{\text{bin}}0$ is the desired limit point OR

Ask the question: For infinitely many k, is it true that $x_k \in \left[0, \frac{1}{2^1}\right)$?

If YES, let

$$\begin{aligned} c_1 &= 0, \\ a_1 &= 0 = 0._{\text{bin}}0, \\ b_1 &= \frac{1}{2} \\ &= a_1 + \frac{1}{2^1}. \\ s_1 &= a_1. \end{aligned}$$

Then

$$b_1 - a_1 = \frac{1}{2^1},$$

$$\text{Infinitely many } x_k \text{ are in } [a_1, b_1).$$

If NO, let

$$\begin{aligned} c_1 &= 1, \\ a_1 &= \frac{1}{2} = 0._{\text{bin}}1, \\ b_1 &= 1 = 1._{\text{bin}}0 \\ &= a_1 + \frac{1}{2^1}. \\ s_1 &= a_1. \end{aligned}$$

Then

$$b_1 - a_1 = \frac{1}{2^1},$$

$$\text{Infinitely many } x_k \text{ are in } [a_1, b_1).$$

Now continue, ...

$$\begin{aligned} x &= \lim_{n\to\infty} s_n \\ &= \lim_{n\to\infty} \left(s_n + \frac{1}{2^n}\right) \end{aligned}$$

Note that $0 \le x - s_n = |x - s_n| \le \frac{1}{2^n}$.