MthT 430 Notes Chap7c Bounded Monotone Sequences Have Limits

BISHL: Bounded Increasing Sequences Have Limits

Theorem. *Let* $\{x_n\}_{n=1}^{\infty}$ *be a bounded monotone increasing sequence; i.e.*

$$x_1 \leq x_2 \leq \ldots,$$

and there is a number M *such that for* $n = 1, 2, \ldots,$

$$x_n \leq M.$$

Then there is a number L *such that*

$$\lim_{n\to\infty} x_n = L.$$

Proof using (P13–BIN): Without loss of generality, we assume that

$$0 \leq x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots < 1.$$

We will construct a binary expansion for L.

A picture is helpful!

Divide the interval $[0, 1)$ into two halves.

Is there an n_1 such that $x_{n_1} \geq m_0 = \frac{1}{2} = 0._{\text{bin}}1$?

If NO, let $c_1 = 0$, $a_1 = 0 = 0._{\text{bin}}c_1$, $b_1 = m_0 = a_1 + \frac{1}{2}$. If YES, let $c_1 = 1$, $a_1 = m_0 = \frac{1}{2} = 0._{\text{bin}}c_1$, $b_1 = a_1 + \frac{1}{2} = 1$.

In both cases, for $n \geq n_1$, $a_1 = 0._{\text{bin}}c_1 \leq x_n \leq b_1 = a_1 + \dfrac{1}{2^1}$ and $b_1 - a_1 = \dfrac{1}{2^1}$.

Next Divide the interval $[a_1, b_1) = \left[0._{\text{bin}}c_1, a_1 + \dfrac{1}{2^1}\right)$ into two halves.

Is there an $n_2 > n_1$ such that $x_{n_2} \geq m_1 = a_1 + \dfrac{1}{2^2}$?

If NO, let $c_2 = 0$, $a_2 = a_1 = 0._{\text{bin}}c_1c_2$, $b_2 = m_1 = a_2 + \dfrac{1}{2^2}$. If YES, let $c_2 = 1$, $a_2 = m_1 = 0._{\text{bin}}c_1c_2$, $b_2 = b_1 = a_2 + \dfrac{1}{2^2}$.

In both cases, for $n \geq n_2$, $a_2 = 0._{\text{bin}}c_1c_2 \leq x_n < b_2 = a_2 + \dfrac{1}{2^2}$ and $b_2 - a_2 = \dfrac{1}{2^2}$.

By recursion (on k), if $n_k > n_{k-1}$, $c_1, \ldots, c_k$, $a_k = 0._{\text{bin}} c_1 \ldots c_k$, $b_k = a_k + \dfrac{1}{2^k}$ have been defined so that for $n \geq n_k$,

$$a_k = 0._{\text{bin}} c_1 \ldots c_k \leq x_n < b_k = a_k + \frac{1}{2^k},$$

divide the interval $[a_k, b_k)$ into two halves.

Is there an $n_{k+1} > n_k$ such that $x_{n_{k+1}} \geq m_k = a_k + \dfrac{1}{2^{k+1}}$?

If NO, let $c_{k+1} = 0$, $a_{k+1} = a_k = 0._{\text{bin}} c_1 c_2 \ldots c_{k+1}$, $b_{k+1} = m_k = a_{k+1} + \dfrac{1}{2^{k+1}}$. If YES, let $c_{k+1} = 1$, $a_{k+1} = m_k = 0._{\text{bin}} c_1 c_2 \ldots c_{k+1}$, $b_{k+1} = b_k = a_{k+1} + \dfrac{1}{2^{k+1}}$.

In both cases, $n_{k+1} > n_k$, $c_1, \ldots, c_k, c_{k+1}$, $a_{k+1} = 0._{\text{bin}} c_1 \ldots c_k c_{k+1}$, $b_{k+1} = a_{k+1} + \dfrac{1}{2^{k+1}}$ have been defined so that for $n \geq n_{k+1}$,

$$a_{k+1} = 0._{\text{bin}} c_1 \ldots c_{k+1} \leq x_n < b_{k+1} = a_{k+1} + \frac{1}{2^{k+1}},$$

Let

$$\begin{aligned} L &= 0._{\text{bin}} c_1 \ldots c_k \ldots \\ &= \lim_{k \to \infty} a_k \\ &= \lim_{k \to \infty} b_k \end{aligned}$$

We have that $L = \lim_{n \to \infty} x_n$ since for all k, and $n > n_k$,

$$0 \leq L - x_n \leq b_k - x_n \leq b_k - a_k \leq \frac{1}{2^k}.$$