# MthT 430 Projects Chap 8b – Intermediate Value Property Remarks

1. Let $f$ be a continuous function on $[0, 1]$ such that
   - $f(0) > 0$,
   - $f(1) < 1$.

   Draw a graph of several (not too complicated) continuous functions, $f$, satisfying the two properties to decide whether there is always an $x$, $0 < x < 1$, such that $f(x) = x$.

   Now prove that there is an $x$, $0 < x < 1$, such that $f(x) = x$.

   See also Spivak, Chapter 7, Problem 11.

   **Remark:** Apply the Intermediate Value Theorem to the function $F(x) = f(x) - x$.

2. Suppose that
   - $f$ and $g$ are continuous functions on $[0, 1]$
   - $f(0) > g(0)$,
   - $f(1) < g(1)$.

   Draw graphs of several pairs of continuous functions, $f$, $g$, satisfying the three properties to decide whether there is always an $x$, $0 < x < 1$, such that $f(x) = g(x)$.

   Now prove that if $f$, $g$, are continuous there is an $x$, $0 < x < 1$, such that $f(x) = g(x)$.

   **Remark:** Apply the Intermediate Value Theorem to the function $F(x) = f(x) - g(x)$.

3. Suppose we are working with a **number system** (such as the rational numbers $\mathbf{Q}$) which satisfies (P1 – P12), but does **not** satisfy (P13–LUB); *id est*, there is a non empty set $A$ of numbers, $A$ is bounded above, but $A$ does not have a least upper bound. For this $A$, let

   $$B_A \equiv \{b \,|\, b \quad \text{is an upper bound for } A.\}$$

   Define

   $$f(x) = \begin{cases} 1, & x \in B_A, \\ -1, & x \notin B_A. \end{cases}$$

   Show that $f$ is continuous at all $x$, but does **not** satisfy the Intermediate Value Property (IVP).

   **Remark:** To show that $f$ is continuous everywhere, consider two cases:

- $b \in B_A$: $f(b) = 1$. Since $b$ is not the least upper bound, there is a $\delta > 0$ such that $b - \delta \in B_A$. Then $f \equiv 1$ on $(b - \delta, \infty)$.

- $b \notin B_A$: Since $b$ is not an upper bound for $A$, There is a $\delta > 0$, such that $b + \delta \in A$. Then $f \equiv -1$ on $(-\infty, b + \delta)$.

Thus NOT (P13–LUB) implies NOT (P13–CFIVP). This shows that (P13–CFIVP) implies (P13–LUB) as stated in chap8b.tex.

See `http://www.math.uic.edu/~lewis/mtht430/chap8b.pdf#CFIVP`

**Remark:** If $A, B_A$, are the sets described in the previous problem, assume that $A$ is a subset of $[0, 1/2)$. We can use a *folding string* argument to construct a nondecreasing sequence of finite binary expansions $s_n = \cdot_{\mathrm{bin}} c_1 \ldots c_n$ such that, for every $n$, $s_n$ is not an upper bound for $A$, but $s_n + (1/2^n)$ is an upper bound for $A$. To do this, at each step select the left interval $[a_n, m_n)$ iff the midpoint $m_n$ is an upper bound for $A$.

It follows that $B_A$ is exactly the set of upper bounds for the sequence $\{s_n\}$. Now conclude that the $\lim_{n\to\infty} s_n$ does not exist *in the number system where $B$ lives*!

Thus NOT (P13–LUB) implies NOT (P13–BIN) and NOT (P13–BISHL). Either (P13–BIN) or (P13–BISHL) implies (P13–LUB).

There are other arguments to show the *equivalence* of (P13–LUB), (P13–CFIVP), (P13–BIN), and (P13–BISHL).