MthT 430 Chapter 9 Derivatives

Definition. *(p. 149) The function f is* differentiable at a *if*

$$\lim_{h\to 0}\frac{f\,(a+h)-f\,(a)}{h}=f'(a) \text{ exists.}$$

N.B. The function f' is a function whose domain is the set of all numbers a such that $f'(a)$ exists.

The *tangent line* to the graph of f at $(a, f(a))$ is the line through $(a, f(a))$ with slope $f'(a)$. By the point–slope form, the equation (formula) for the tangent line [to the graph of f at $(a, f(a))$] is

$$y = T_a(x) = f(a) + f'(a)\,(x-a)\,.$$

If f is differentiable at a, the *error of the tangent line approximation* is

$$\begin{aligned}\phi_a(x) &\equiv f(x) - T_a(x),\\ &= f(x) - \left(f(a) + f'(a)\,(x-a)\right).\end{aligned}$$

Note that

$$\lim_{x\to a}\frac{\phi_a(x)}{x-a}=0.$$

Theorem 1. *If f is differentiable at a, then f is continuous at a.*

Proof.

$$\begin{aligned}\lim_{h\to 0}\left(f(a+h)-f(a)\right) &= \lim_{h\to 0}\frac{f\,(a+h)-f\,(a)}{h}\cdot h\\ &= f'(a)\cdot 0\\ &= 0.\end{aligned}$$

Examples

1. $f(x) = |x|$. f is continuous (at all points in its domain).

$$f'(a) = \begin{cases} 1, & a>0,\\ -1, & a<0,\\ \text{undefined}, & a=0.\end{cases}$$

2. There are functions f which are continuous everywhere, but differentiable nowhere.

3. For $n = 1, 2, \ldots$, the *power–n function*, $\mathrm{Power}_n(x) = x^n$, is differentiable (everywhere) and

$$\mathrm{Power}_n'(x) = nx^{n-1}.$$

4. We will assume that the trigonometric functions, sin and cos, are differentiable and

$$\begin{aligned} \sin'(x) &= \cos(x), \\ \cos'(x) &= -\sin(x). \end{aligned}$$

5. Let

$$f(x) = \begin{cases} x\sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then f is continuous at 0, but not differentiable at 0. The difference quotient at 0 is

$$\begin{aligned} \frac{f(0+h) - f(0)}{h} &= \frac{h\sin(1/h)}{h} \\ &= \sin(1/h), \end{aligned}$$

which does not have a limit as $h \to 0$.

6. Let

$$f(x) = \begin{cases} x^2\sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then f is continuous at 0, and differentiable at 0. The difference quotient at 0 is

$$\begin{aligned} \frac{f(0+h) - f(0)}{h} &= \frac{h^2\sin(1/h)}{h} \\ &= h\sin(1/h) \\ &\to 0 \text{ as } h \to 0. \end{aligned}$$