MthT 491 Distributive Properties and Negative Numbers

To emphasize the important role of the distributive property in dealing with positive and negative numbers, we construct a system of **Numbers**, [weird] numbers, which

- satisfies all the properties of an ordered field except for the distributive property:

P9 For all a, b, c,

$$a \text{ times } (b + c) = (a \text{ times } b) + (a \text{ times } c) = a \text{ times } b + a \text{ times } c.$$

- the product of two [weird] negative numbers is a [weird] negative number.

For the time being we will denote the numbers we are using to by **Numbers**. We shall list the primitive properties – that is, develop a minimal list of properties from which results can be deduced.

We shall assume there is a set **Numbers**, with binary operations $+$ (plus, addition) and $\cdot$ (times, multiplication) defined.

We start with a *Commutative Group*, $(G, +)$ – a set of numbers G, with a binary operation $+$ (plus, addition), which satisfies

Properties of $+$

P1 For all a, b, c, in G,

$$a + (b + c) = (a + b) + c$$

P2 There is a number 0 in G such that for all a,

$$a + 0 = 0 + a = a.$$

P3 For all a, there is a number $-a$ such that

$$a + (-a) = (-a) + a = 0.$$

P4 For all a, b,

$$a + b = b + a.$$

Examples include

- $\mathbf{Z}$, the set of all integers.

- **R**, the set of real numbers.
- **Q**, the set of rational numbers.
- **C**, the set of complex numbers.
- $\mathbf{Z} + i\mathbf{Z}$, the set of "complex integers."

Temporarily, we will assume

- G is nontrivial in the sense that there is an element $U \in G$, $U \neq 0$.

We now define weird multiplication, $\star$, on G by

For all $a, b \in G$,

$$a \star b \equiv a + b - U.$$

Properties of $\star$

P5 For all a, b, c,

$$a \star (b \star c) = (a \star b) \star c$$

Proof.

$$\begin{aligned} a \star (b \star c) &= a + (b + c - U) - U \\ &= \ldots \\ &= (((a + b) - U) + c) - U \\ &= ((a \star b) + c) - U \\ &= (a \star b) \star c. \end{aligned}$$

P6 There is a number $1 \neq 0$ such that for all a,

$$a \star 1 = 1 \star a = a.$$

Proof. Let $1 \equiv U$.

$$\begin{aligned} 1 \star a &= U + a - U \\ &= a \\ &= a \star U. \end{aligned}$$

P7 For all $a \neq 0$, there is a number a^{-1} such that

$$a \star (a^{-1}) = (a^{-1}) \star a = 0.$$

Proof. For any a, let

$$\begin{aligned} a^{-1} &\equiv -a + U + U, \\ a \star a^{-1} &= a + (-a + U + U) - U \\ &= U \end{aligned}$$

P8 For all a, b,

$$a \star b = b \star a.$$

N.B. With the multiplication $\star$,

$$\begin{aligned} 0 \star 0 &= 0 + 0 - U \\ &= -U \\ &\neq 0. \\ (-U) \star (-U) &= -U - U - U. \end{aligned}$$

If $U \neq -U$,

$$\begin{aligned} -(0 \star 0) &= -(-U) \\ &= U \\ &\neq (-0) \star 0 \\ &= 0 * 0 \\ &= -U. \end{aligned}$$

The structure $(G, +, \star)$ satisfies all the properties of a *field*, except the glue which relates multiplication and addition, the *distributive property*:

Property of $\cdot$ with $+$

P9 For all a, b, c,

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c) = a \cdot b + a \cdot c.$$

Positive Numbers and Order

Within our set of numbers, we say that a collection of numbers, P, is a *positive set*, or a set of *positive numbers* if P satisfies P10 – P12:

P10 For every a, one and only one of the following holds:

(i) $a = 0$,

(ii) a is in the collection P,

(iii) a is in the collection P.

P11 If a and b are in the collection P, then $a + b$ is in the collection P.

P12 If a and b are in the collection P, then the product of a and b is in the collection P.

If P is a given positive set, we define inequalities or P–inequalities by:

$$a < b \, (a <_{\mathcal{P}} b) \text{ iff } b - a \in P.$$

Weird Example $(Z, +, \star)$

As an example, we consider $(Z, +, \star)$, $U = 1$, the usual "1". The system

- Satisfies P1 – P8.
- Does **not** satisfy P9. Give a counterexample!
- $0 \star 0 \neq 0$.
- There are nonzero a and b such that $a \star b = 0$. Give examples.
- The set can be be *ordered* in such a way that "1" is not positive.

In our example $(\mathbf{Z}, +, \star)$, we take as a *weird positive set*

$$\mathcal{P}_\star = \{-1, -2, \ldots\},$$

the usual set of negative integers. The *weird negative* integers are

$$\mathcal{N}_\star = \{1, 2, \ldots\},$$

the usual set of positive integers.

We have P10 (trichotomy).

Now verify P11 and P12. A typical element of $\mathcal{P}_\star$ is of the form $-a$, with a a usual positive integer. If $(-a)$, $(-b)$, are in $\mathcal{P}_\star$, then

$$\begin{aligned}(-a) + (-b) &= -(a+b) \in \mathcal{P}_\star, \\ (-a) \star (-b) &= -(a+b) - 1 \\ &= -(a+b+1) \in \mathcal{P}_\star.\end{aligned}$$

Here the *weird product* of two *weird negative* integers is always *weird negative*: For a and b *weird negative*, i.e., usual positive integers

$$a \star b = a + b - 1$$

is a usual positive integer, i.e., *weird negative*.

More Examples

We consider the even and odd integers. We know that

$$\begin{aligned} \text{odd} + \text{even} &= \text{odd}, \\ \text{even} + \text{even} &= \text{even}, \\ \text{odd} \cdot \text{odd} &= \text{odd}, \\ \text{odd} \cdot \text{even} &= \text{even}. \end{aligned}$$

Thus the role of zero for addition is played by *even*.

We construct the addition table:

+ (plus)	odd	even
odd	even	
even		

The usual multiplication table is:

· (times)	odd	even
odd		
even		

The weird multiplication table is:

⋆ (weird)	odd	even
odd	even	odd
even	odd	even

Note that

$$\begin{aligned} \text{odd} \star (\text{even} + \text{odd}) &= \text{even}, \\ (\text{odd} \star \text{even}) + (\text{odd} \star \text{odd}) &= \text{odd} + \text{even} \\ &= \text{odd}. \end{aligned}$$

N.B. With the usual addition, there is no way to define to define a *positive set* which satisfies P10 and P11.