Math 300, Spring 2004 **Writing for Mathematics** **Monday Sections**

The Mathematics for Essay 2

The purpose of these notes are to explain some of the mathematics behind Essay 2. Your own essay should not just repeat these arguments but have a more geometric flavor. Write about how you can physically place the blocks. You may assume basic facts about geometric sums and series. Let r be any real number and let n be a non-negative integer. The sum

$$1 + r + r^2 + \cdots + r^n \tag{1}$$

is a *geometric sum* and the infinite series

$$1 + r + r^2 + \cdots + r^n + \cdots \tag{2}$$

is a *geometric series.*

Suppose further that $r \neq 1$. Then the geometric sum (1) can be computed by the formula

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}.$$

This fact, which you may assume, is easily proved proved by mathematical induction.

Now suppose that $|r| < 1$. Then $\lim_{n \longrightarrow \infty} r^n = 0$ which means the geometric series (2) converges to $\dfrac{1}{1-r}$ by the preceding equation. We write

$$1 + r + r^2 + \cdots + r^n + \cdots = \frac{1}{1-r} \tag{3}$$

to indicate that the series converges and to designate the limit of the sequence of partial sums.

Your essay will involve the geometric series

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \cdots. \tag{4}$$

Since $|\frac{1}{2}| < 1$, it follows by (3) that (4) converges and $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \cdots = 2$. The Deluxe blocks are cubes with side lengths 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{5}$, ... Your essay involves analyzing the sum of their side lengths

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \cdots + \frac{1}{16} + \cdots.$$

The preceding series is called the *harmonic series*. Think of the terms of the geometric series (4) as markers for grouping terms of the harmonic series as follows:

$$\mathbf{1} + \mathbf{\frac{1}{2}} + (\frac{1}{3} + \mathbf{\frac{1}{4}}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \mathbf{\frac{1}{8}}) + (\frac{1}{9} + \cdots + \mathbf{\frac{1}{16}}) + \cdots. \tag{5}$$

We will find an overestimate and an underestimate for the sum of the terms in each of the parenthesized groups. You will see a pattern emerging in our calculations:

$$1 = \frac{1}{2} + \frac{1}{2} > \frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2},$$

$$1 = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} > \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2},$$

$$1 = \frac{1}{8} + \cdots + \frac{1}{8} = 8(\frac{1}{8}) > \frac{1}{9} + \cdots + \frac{1}{16} > \frac{1}{16} + \cdots + \frac{1}{16} = 8(\frac{1}{16}) = \frac{1}{2},$$

$$\vdots$$

Using (5) and our underestimates, we see that

$$\begin{aligned} & 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{8} + \frac{1}{9} + \cdots + \frac{1}{16} + \cdots \\ = \quad & 1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) + (\frac{1}{9} + \cdots + \frac{1}{16}) + \cdots \\ > \quad & 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots. \end{aligned}$$

Thus the partial sums of the harmonic series grow without bound which is expressed by

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \infty.$$

Below is a formal proof of the the fact that the sums of terms in parenthesized groupings lie between $\frac{1}{2}$ and 1. You should *not* include the proof in your essay; the mathematics of your essay is to be treated informally. Observe that the terms of a parenthesized group are given by $\frac{1}{2^n+1}, \ldots, \frac{1}{2^{n+1}}$ for some $n \geq 1$.

Lemma 1 *Let n be a positive integer. Then* $\frac{1}{2} < \frac{1}{2^n+1} + \cdots + \frac{1}{2^{n+1}} < 1$.

Proof: Since $2^{n+1} = 2^n + 2^n$ the sum in the statement of the lemma has 2^n terms. Each term has the form $\frac{1}{2^n+\ell}$ for some $1 \leq \ell \leq 2^n$ and thus satisfies

$$\frac{1}{2^{n+1}} \leq \frac{1}{2^n+\ell} \leq \frac{1}{2^n}.$$

At least one of the terms is larger than $\frac{1}{2^{n+1}}$ and one at least one is smaller than $\frac{1}{2^n}$. Therefore

$$\frac{1}{2} = 2^n(\frac{1}{2^{n+1}}) < \frac{1}{2^n+1} + \cdots + \frac{1}{2^{n+1}} < 2^n(\frac{1}{2^n}) = 1.$$

□

JTB 02/17/09 adapted from David Radford.