Essay 2: Geometry of Cubes

Math 300, Fall 2018, Schneider

1 Content

In this essay you will systematically construct a coherent mathematical theory of N-dimensional cubes. You will formally define these objects using $\mathbb{R}^n$ coordinates and prove some enumerative formulas about their structure. Be sure to give concrete examples in dimensions 1, 2, & 3 via graphical illustrations, on which the reader will base their higher-dimensional intuition.

Please define, explain, and illustrate the notion of an N-cube and its n-elements (vertices, edges, faces, cells, etc.). Prove that the elements of an N-cube are themselves smaller cubes. Justify this observation using your definition. Then, give both recursive and explicit formulas for the number of n-elements. Give a table of values for the element-counts, and prove your formulas using explicit reference to your definitions.

2 Style

Your audience is at least as mathematically sophisticated as yourself. Write in the style of an advanced college math textbook or a research article. I will be looking for well-organized exposition of a coherent mathematical theory. Motivate the theory, define your terms, and prove your theorems. Your proofs should explicitly use your definitions.

Assume your reader is familiar with general properties of $\mathbb{R}^n$, such as its coordinate system, the notion of convexity, and the dimension of a convex set. All the math is presented in class lectures, as well as wikipedia and the notes below.

3 Math

These notes are intentionally condensed, abridged, and abbreviated. Expand them into human-readable form in your essay.

3.1 Definitions

The **standard** N**-cube** is the subset of $\mathbb{R}^N$ whose vertices are of the form $(x_1, \ldots, x_N)$ with each $x_i = 0$ or 1.

Any figure in $\mathbb{R}^m$ $(m \geq N)$ that is geometrically similar to the standard N-cube is an N**-cube**.

In the standard N-cube, an n**-element's** vertices are precisely those that match each other in exactly $N - n$ of their coordinates $(x_1, \ldots, x_N)$.

3.2 Formulas

Define a function

$$C(N, n) = \text{number of } n\text{-elements in an } N\text{-cube.}$$

Demonstrate and prove this recursive formula:

$$\begin{aligned} C(N, n) &= 2 \cdot C(N-1, n) + C(N-1, n-1) \\ C(N, 0) &= 2^N \\ C(0, n) &= 0 \quad (n \geq 1) \end{aligned}$$

and this explicit formula:

$$C(N, n) = 2^{N-n} \cdot \binom{N}{n}.$$

3.3 Table

You should include a table of values. Here's a template.

		Little cube (n)					
		0	1	2	3	4	5
	0						
Big	1						
cube	2						
(N)	3						
	4						
	5						

3.4 Proofs

You have several theorems to state and prove. Here are a few to get you started.

Theorem 1
The n-elements of the standard N-cube are themselves n-cubes.

Proof. In the standard N-cube, the vertices of a n-element agree in $(N-n)$ coordinates. Omitting these coordinates produces the vertices of the standard n-cube. $\square$

Theorem 2
The number of n-elements on an N-cube is given by

$$C(N,n) \;=\; 2^{N-n}\cdot\binom{N}{n}$$

Proof. (Presented in lecture)

Theorem 3
The number of n-elements on an N-cube satisfies

$$\begin{aligned} C(N,0) &= 2^N \\ C(0,n) &= 0 \quad (n \geq 1) \\ C(N,n) &= 2\cdot C(N-1,n) + C(N-1,n-1) \end{aligned}$$

Proof. (Presented in lecture)