Generalized Hypothesis Testing and Maximizing the Success Probability in Financial Markets

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Abstract. We study the generalized composite pure and randomized hypothesis testing problems. In addition to characterizing the optimal tests, we examine the conditions under which these two hypothesis testing problems are equivalent, and provide counterexamples when they are not. This analysis is useful for portfolio optimization to maximize some success probability given a fixed initial capital. The corresponding dual is related to a pure hypothesis testing problem which may or may not coincide with the randomized hypothesis testing problem. Our framework is applicable to both complete and incomplete market settings.

1 Introduction

In a financial market, suppose the smallest super-hedging price for a contingent claim F of expiration T is given by F_0. Then, given the initial capital $x < F_0$, how can one construct a strategy π in order to maximize the probability that the terminal wealth $X^{x,\pi}(T)$ exceeds F? A related question is: what is the minimum initial capital required to ensure a given success probability $\mathbb{P}\{X^{x,\pi}(T) \geq F\}$? Furthermore, the claim F can be more generally viewed as a benchmark to beat, then the successful event is based on whether the terminal portfolio value will outperform the given benchmark (see [1]).

These problems have been studied in the framework of quantile hedging, dating back to Föllmer and Leukert [4]. As is well known, the quantile hedging problem can be formulated as a pure hypothesis testing problem, and Neyman-Pearson Lemma provides a characterization of the solution.

Although maximizing the success probability $\mathbb{P}\{X^{x,\pi}(T) \geq F\}$ in this work is equivalent to minimizing the shortfall risk (see [2, 5, 7, 9]) $\mathbb{P}\{X^{x,\pi}(T) < F\} = \rho(-(F - X^{x,\pi}(T))^+)$ with $\rho(X) = \mathbb{P}\{X < 0\}$, this risk measure $\rho(\cdot)$ does not satisfy either convexity or continuity. Consequently, the value function of such a problem is only equivalent to a pure hypothesis testing problem. In fact, we provide an example (see Example 1) to show that it may be strictly smaller than the randomized hypothesis testing problem.

The main contribution of the current paper is the analysis of the generalized composite pure and randomized hypothesis testing problems (see (2) and (9)). The sets of alternative hypotheses $\mathcal{G}$ and null hypotheses $\mathcal{H}$ are not limited to probability density functions but simply non-negative L^1-bounded random variables. This is applicable to statistical hypothesis testing with non-constant significance levels. In addition to solving both problems, we investigate when a pure test solves both the randomized and pure testing problems. To this end, we provide sufficient conditions that make the two hypothesis testing problems equivalent. Moreover, we show the convexity and continuity of the value function for the randomized hypothesis testing problem (see Theorem 2).

Our analysis is used to study the portfolio choice to maximize the success probability in an incomplete market. This problem is equivalent to a composite pure hypothesis testing problem, which may be strictly dominated by the related randomized testing problem (see Example 2). We provide the sufficient conditions for these two problems to coincide (see Theorem 4). Finally, we provide examples with explicit solutions for maximizing the success probability in a complete market and in an incomplete stochastic volatility market.

2 Generalized Composite Hypothesis Testing

In the background, we fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Denote by $\mathbb{E}[\,\cdot\,]$ the expectation under $\mathbb{P}$, and $L^{1,+}$ the class of all non-negative $\mathcal{F}$-measurable random variables X satisfying $\mathbb{E}[X] < \infty$. The pure tests and randomized tests are represented by the two collections of random variables, respectively, $\mathcal{X} = \{X : \Omega/\mathcal{F} \mapsto [0,1]/\mathcal{B}([0,1])\}$ and $\mathcal{I} = \{X : \Omega/\mathcal{F} \mapsto \{0,1\}/2^{\{0,1\}}\}$. In addition, $\mathcal{G}$ and $\mathcal{H}$ are two given collections of $\mathbb{P}$-integrable non-negative $\mathcal{F}$-measurable random variables.

Let us consider the randomized composite hypothesis testing problem:

$$\sup_{X \in \mathcal{X}} \inf_{G \in \mathcal{G}} \mathbb{E}[GX] \quad \text{subject to} \quad \sup_{H \in \mathcal{H}} \mathbb{E}[HX] \leq x. \tag{1}$$

This problem is equivalent to

$$V(x) := \sup_{X \in \mathcal{X}_x} \inf_{G \in \mathcal{G}} \mathbb{E}[GX] \tag{2}$$

where $\mathcal{X}_x := \{X \in \mathcal{X} : \mathbb{E}[HX] \leq x,\ \forall H \in \mathcal{H}\}$ is the set of all feasible randomized tests.

This randomized hypothesis testing problem is similar to that studied by Cvitanic and Karatzas [3], except that G and H in (1) and (2) are not necessarily Radon-Nikodym derivatives. This slight generalization allows $\|H\|_{L^1}$ to be different from each other among $\mathcal{H}$, which is applicable to statistical hypothesis testing with non-constant significance levels; see also Remark 5.2 by Rudloff and Karatzas [8]. Similar to [3], we make the following standing assumption:

Assumption 1. *Assume that $\mathcal{G}$ and $\mathcal{H}$ are bounded in $L^{1,+}$, and $\mathcal{G}$ is convex and closed under $\mathbb{P}$-a.e. convergence.*

Define the set $\mathcal{H}_x := \{H \in L^{1,+} : \mathbb{E}[HX] \leq x,\ \forall X \in \mathcal{X}_x\}$. It is straightforward to check that $\mathcal{H}_x$ is a convex set containing convex hull of $\mathcal{H}$. The following theorem gives the characterization of the solution for (2).

Theorem 2. *Under Assumption 1, there exists* $(\hat{G},\hat{H},\hat{a},\hat{X})\in\mathcal{G}\times\mathcal{H}_x\times[0,\infty)\times\mathcal{X}_x$ *satisfying*

$$\hat{X}=I_{\{\hat{G}>\hat{a}\hat{H}\}}+BI_{\{\hat{G}=\hat{a}\hat{H}\}},\ \textit{for some } B:\Omega/\mathcal{F}\mapsto[0,1]/\mathcal{B}([0,1]), \tag{3}$$

$$\mathbb{E}[\hat{H}\hat{X}]=x, \tag{4}$$

and

$$\mathbb{E}[\hat{G}\hat{X}]\le\mathbb{E}[G\hat{X}],\quad\forall G\in\mathcal{G}. \tag{5}$$

Then, $V(x)$ *of* (2) *is given by*

$$V(x)=\mathbb{E}[\hat{G}\hat{X}]=\inf_{a\ge0}\{xa+\inf_{\mathcal{G}\times\mathcal{H}}\mathbb{E}[(G-aH)^+]\}, \tag{6}$$

which is continuous, concave, and non-decreasing in $x\in[0,\infty)$.

Furthermore, $(\hat{G},\hat{H})$ *and* $(\hat{G},\hat{H},\hat{a})$ *respectively attain the infimum of*

$$\mathbb{E}[(G-\hat{a}H)^+],\ \textit{and}\ xa+\inf_{\mathcal{G}\times\mathcal{H}}\mathbb{E}[(G-aH)^+]. \tag{7}$$

Proof. The proof of (6) and (7), and the existence of the associated $(\hat{G},\hat{H},\hat{a},\hat{X})$ under Assumption 1 can be completed following the procedures in [3], by replacing their two probability density sets to the $L^{1,+}$-bounded sets $\mathcal{G}$ and $\mathcal{H}$ here. Monotonicity of $V(x)$ follows from its definition. For any $x_1,x_2\ge0$,

$$\tfrac{V(x_1)+V(x_2)}{2}\le\inf_{a\ge0,(G,H)\in\mathcal{G}\times\mathcal{H}}\mathbb{E}\left[\tfrac{1}{2}(x_1+x_2)a+(G-aH)^+\right]=V\left(\tfrac{x_1+x_2}{2}\right)$$

implies the concavity of $V(x)$. The boundedness together with the concavity yields the continuity.

Another difference lies in $\hat{a}$, which belongs to $[0,\infty)$ in Theorem 2, as opposed to $(0,\infty)$ by Proposition 3.1 and Lemma 4.3 in Cvitanic and Karatzas [3].

We proceed to give a simple explicit construction of B in (3) of Theorem 2.

Corollary 1. *Let* $(\hat{G},\hat{H},\hat{a},\hat{X})\in\mathcal{G}\times\mathcal{H}_x\times[0,\infty)\times\mathcal{X}_x$ *be given by Theorem 2. Then,* B *in* (3) *can be assigned as follows:*

(i) If $\mathbb{E}[\hat{H}I_{\{\hat{G}>\hat{a}\hat{H}\}}]=x$, *then* $B=0$.
(ii) If $\mathbb{E}[\hat{H}I_{\{\hat{G}\ge\hat{a}\hat{H}\}}]=x>\mathbb{E}[\hat{H}I_{\{\hat{G}>\hat{a}\hat{H}\}}]$, *then* $B=1$.
(iii) If $\mathbb{E}[\hat{H}I_{\{\hat{G}\ge\hat{a}\hat{H}\}}]>x>\mathbb{E}[\hat{H}I_{\{\hat{G}>\hat{a}\hat{H}\}}]$, *then* $B=\frac{x-\mathbb{E}[\hat{H}I_{\{\hat{G}>\hat{a}\hat{H}\}}]}{\mathbb{E}[\hat{H}I_{\{\hat{G}=\hat{a}\hat{H}\}}]}>0$.

To summarize, the randomized test $\hat{X}$ given by (3) in Theorem 2 with the choice of B in Corollary 1 solves the generalized composite hypothesis testing problem (2). In Corollary 1, B is simply a constant in each scenario (i)-(iii), and the resulting $\hat{X}$ is an indicator under (i) and (ii) but not (iii).

However, we emphasize that there may exist alternative ways to specify B as the solution. In particular, if it turns out that B can be assigned as an indicator function (even under scenario (iii)), then $\hat{X}$ will also be an indicator, and therefore, a *pure* test! This leads to the interesting question: when does a pure test solve the randomized composite hypothesis testing problem?

Motivated by this, we define the pure composite hypothesis testing problem:

$$\sup_{X\in\mathcal{I}}\inf_{G\in\mathcal{G}}\mathbb{E}[GX] \quad \text{subject to} \quad \sup_{H\in\mathcal{H}}\mathbb{E}[HX]\leq x. \tag{8}$$

This is equivalent to solving

$$V_1(x)=\sup_{X\in\mathcal{I}_x}\inf_{G\in\mathcal{G}}\mathbb{E}[GX], \tag{9}$$

where $\mathcal{I}_x:=\{X\in\mathcal{I}:\mathbb{E}[HX]\leq x,\ \forall H\in\mathcal{H}\}$ consists of all candidate pure tests.

From their definitions, we see that $V(x)\geq V_1(x)$. However, one cannot expect $V_1(x)=V(x)$ in general, as demonstrated in the next example.

Example 1. Fix $\Omega=\{0,1\}$ and $\mathcal{F}=2^\Omega$, with $\mathbb{P}\{0\}=\mathbb{P}\{1\}=1/2$. Define the collections $\mathcal{G}=\{G:G(0)=G(1)=1\}$, and $\mathcal{H}=\{H:H(0)=1/2,H(1)=3/2\}$. In this simple setup, direct computations yield that

1. For the randomized hypothesis testing, $V(x)$ is given by

$$V(x)=\begin{cases}\mathbb{E}[4xI_{\{0\}}]=2x, & \text{if}\ \ 0\leq x<1/4;\\ \mathbb{E}[I_{\{0\}}+\frac{4x-1}{3}I_{\{1\}}]=\frac{2x+1}{3}, & \text{if}\ \ 1/4\leq x<1;\\ \mathbb{E}[1]=1, & \text{if}\ \ x\geq 1.\end{cases} \tag{10}$$

2. For the pure hypothesis testing, $V_1(x)$ is given by

$$V_1(x)=\begin{cases}\mathbb{E}[0]=0, & \text{if}\ \ 0\leq x<1/4;\\ \mathbb{E}[I_{\{0\}}]=\frac{1}{2}, & \text{if}\ \ 1/4\leq x<1;\\ \mathbb{E}[1]=1, & \text{if}\ \ x\geq 1.\end{cases} \tag{11}$$

It follows that $V_1(x)<V(x)$, $\forall x\geq 0$. In fact, $V_1(x)$ does not preserve concavity and continuity, while $V(x)$ does. $\square$

If there is a pure test that solves both the pure and randomized composite hypothesis testing problems, then we must have the equality $V_1(x)=V(x)$. An important question is when this phenomenon of equivalence occurs. In search of a sufficient condition, we infer from (3) that if $\mathbb{P}\{G=aH\}=0$ for all $(G,H,a)\in\mathcal{G}\times\mathcal{H}\times[0,\infty)$, then we obtain $\hat{X}=I_{\{\hat{G}>\hat{a}\hat{H}\}}$, which is a pure test. The following lemma provides another sufficient condition (see (12) below) for the equivalent between V_1 and V.

Lemma 1. *Suppose $\mathcal{H}=\{H\}$ is a singleton, and we assume there exists a $\mathcal{F}$-measurable random variable Y, such that the function*

$$g(y)=\mathbb{E}[HI_{\{Y<y\}}],\ \forall y\in\mathbb{R} \tag{12}$$

is continuous. Then there exists a pure test $\hat{X}$ that solves both problems (2) *and* (9)*. As a result, $V_1(x)=V(x)$.*

Proof. Let $(\hat{G}, \hat{H}, \hat{a}, \hat{X}) \in \mathcal{G} \times \mathcal{H} \times [0, \infty) \times \mathcal{X}_x$ be given by Theorem 2. Since $\mathcal{H} = \{H\}$ is a singleton, then $\hat{H} = H$ follows from the definition of the hypothesis testing problem. We will show the existence of the pure test $\hat{X}$ under (12).

By Corollary 1, we only need to prove the result when $\mathbb{E}[\hat{H} I_{\{\hat{G} \geq \hat{a}\hat{H}\}}] > x > \mathbb{E}[\hat{H} I_{\{\hat{G} > \hat{a}\hat{H}\}}]$. Define a function $g_1(\cdot)$ by $g_1(y) = \mathbb{E}[\hat{H} I_{\{\hat{G} = \hat{a}\hat{H}\} \cap \{Y < y\}}]$. Note that $g_1(\cdot)$ is right-continuous since, for any $y \in \mathbb{R}$,

$$
\begin{aligned}
|g_1(y + \varepsilon) - g_1(y)| &= \mathbb{E}[\hat{H} I_{\{\hat{G} = \hat{a}\hat{H}\} \cap \{y \leq Y < y + \varepsilon\}}] \\
&\leq \mathbb{E}[\hat{H} I_{\{y \leq Y < y + \varepsilon\}}] = g(y + \varepsilon) - g(y) \to 0, \quad \text{as } \varepsilon \to 0^+
\end{aligned}
$$

by the continuity of $g(\cdot)$. Similar arguments show that $g_1(\cdot)$ is also left-continuous. Also, observe that

$$
\lim_{y \to -\infty} g_1(y) = 0, \text{ and } \lim_{y \to \infty} g_1(y) = \mathbb{E}[\hat{H} I_{\{\hat{G} = \hat{a}\hat{H}\}}] > x - \mathbb{E}[\hat{H} I_{\{\hat{G} > \hat{a}\hat{H}\}}].
$$

Therefore, there exists $\hat{y} \in \mathbb{R}$ satisfying $g_1(\hat{y}) = x - \mathbb{E}[\hat{H} I_{\{\hat{G} > \hat{a}\hat{H}\}}]$. Now, we can simply set $B = I_{\{Y < \hat{y}\}}$, which is $\mathcal{F}$-measurable. In turn, this yields $\hat{X}$ of (3) as an indicator of the form

$$
\hat{X} = I_{\{\hat{G} = \hat{a}\hat{H}\} \cap \{Y < \hat{y}\} \cup \{\hat{G} > \hat{a}\hat{H}\}}. \tag{13}
$$

By the choice of $\hat{y}$, one can directly verify that the above $\hat{X}$ belongs to $\mathcal{X}_x$ and satisfies (3), (4), and (5).

Next, we summarize two sufficient conditions, which are amenable for the verification.

Theorem 3. *Suppose one of following conditions is satisfied:*

1. *$\mathcal{H} = \{H\}$ is a singleton and there exists a $\mathcal{F}$-measurable random variable with continuous cumulative distribution function with respect to $\mathbb{P}$.*
2. *$\mathbb{P}\{G = aH\} = 0$ for all $(G, H, a) \in \mathcal{G} \times \mathcal{H} \times [0, \infty)$.*

Then $V_1(x) = V(x)$, and there exists an indicator function $\hat{X}$ that solves problems (2) and (9) simultaneously. Furthermore, $x \mapsto V_1(x)$ is continuous, concave, and non-decreasing.

Note that condition 1 in Theorem 3 is slightly stronger than (12). In the next section, we will be apply our analysis of the generalized composite hypothesis testing to quantile hedging in incomplete financial markets.

3 Portfolio to Maximize the Success Probability

We now discuss the portfolio optimization problem whose objective is to maximize the probability of outperforming a (random) benchmark. This related to the well-known problem of quantile hedging introduced by Föllmer and Leukert

[4]. Apply our preceding analysis and the generalized Neyman-Pearson lemma, we will examine our problem in both complete and incomplete markets.

We fix $T > 0$ as the investment horizon and let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ be a filtered complete probability space satisfying the usual conditions. We model a stock price by a $\mathcal{F}_t$-adapted non-negative semi-martingale process $(S(t))_{t\geq 0}$. For notational simplicity, we assume zero risk-free interest rate.

The class of Equivalent Martingale Measures (EMMs), denoted by $\mathcal{Q}$, consists of all probability measures $\mathbb{Q} \sim \mathbb{P}$ on $\mathcal{F}_T$ such that the stock price $S(t)$ is $\mathbb{Q}$ martingale. The associated set of Radon-Nikodym densities is

$$\mathcal{Z} := \Big\{ \frac{d\mathbb{Q}}{d\mathbb{P}} : \mathbb{Q} \in \mathcal{Q} \Big\}.$$

In the classical no-arbitrage pricing theory, these are the candidate pricing measures used for derivative valuation. We require that $\mathcal{Q}$ be non-empty to preclude arbitrage opportunities.

Given an initial capital x and a self-financing strategy π, the investor's trading wealth process satisfies

$$X^{x,\pi}(t) = x + \int_0^t \frac{\pi(u)}{S(u)} dS(u). \tag{14}$$

The trading strategy π belongs to the space of all admissible strategies $\mathcal{A}(x)$ defined by $\mathcal{A}(x) = \{\pi : X^{x,\pi}(t) \geq 0, \ \forall t \in [0,T], \ \mathbb{P} - a.s.\}$. For any random terminal payoff $F \in \mathcal{F}_T$, the smallest super-hedging price (see, for example, Touzi, [4]) is given by

$$F_0 := \sup_{Z\in\mathcal{Z}} \mathbb{E}[ZF]. \tag{15}$$

We can interpret F as the benchmark. In quantile hedging, F is the terminal contingent claim In other words, F_0 is the smallest capital needed for $\mathbb{P}\{X^{x,\pi}(T) \geq F\} = 1$ for some strategy $\pi \in \mathcal{A}(x)$. Note that with less initial capital $x < F_0$ the success probability $\mathbb{P}\{X^{x,\pi}(T) \geq F\} < 1$ for all $\pi \in \mathcal{A}(x)$.

Our objective is to maximize over all admissible trading strategies the success probability with $x < F_0$. Specifically, we solve the optimization problem:

$$\widetilde{V}(x) := \sup_{x_1\leq x} \sup_{\pi\in\mathcal{A}(x_1)} \mathbb{P}\{X^{x_1,\pi}(T) \geq F\} = \sup_{\pi\in\mathcal{A}(x)} \mathbb{P}\{X^{x,\pi}(T) \geq F\}, \qquad x \geq 0. \tag{16}$$

Next, we show that (16) admits a dual representation as a pure hypothesis testing problem.

Proposition 1. *The value function $\widetilde{V}(x)$ of* (16) *is equal to the solution of a pure hypothesis testing problem, that is, $\widetilde{V}(x) = V_1(x)$ where*

$$\begin{cases} V_1(x) = \sup\limits_{A} \mathbb{P}(A) \\ \qquad\qquad \text{over } A \in \mathcal{F}_T \text{ subject to } \sup\limits_{Z\in\mathcal{Z}} \mathbb{E}[ZFI_A] \leq x. \end{cases} \tag{17}$$

Furthermore, if there exists $\hat{A} \in \mathcal{F}_T$ that solves (17), then $\widetilde{V}(x) = \mathbb{P}(\hat{A})$, and the associated optimal strategy π^* is a super-hedging strategy with $X^{x,\pi^*}(T) \geq F I_{\hat{A}}$ $\mathbb{P}$-a.s.

Proof. First, if we set $\mathcal{H} = \{ZF : Z \in \mathcal{Z}\}$ and $\mathcal{G} = \{1\}$, then the right-hand side of (17) resembles the pure hypothesis testing problem in (9).

1. First, we prove that $V_1(x) \geq \widetilde{V}(x)$. For an arbitrary $\pi \in \mathcal{A}(x)$, define the success event $A^{x,\pi} := \{X^{x,\pi}(T) \geq F\}$. Then, $\sup_{Z\in\mathcal{Z}} \mathbb{E}[ZFI_{A^{x,\pi}}]$ is the smallest amount needed to super-hedge $FI_{A^{x,\pi}}$. By the definition of $A^{x,\pi}$, we have that $X^{x,\pi}(T) \geq FI_{A^{x,\pi}}$, i.e. the initial capital x is sufficient to super-hedge $FI_{A^{x,\pi}}$. This implies that $A^{x,\pi}$ is a candidate solution to V_1 since the constraint $x \geq \sup_{Z\in\mathcal{Z}} \mathbb{E}[ZFI_{A^{x,\pi}}]$ is satisfied. Consequently, for any $\pi \in \mathcal{A}(x)$, we have $V_1(x) \geq \mathbb{P}(A^{x,\pi})$. Since $\widetilde{V}(x) = \sup_{\pi\in\mathcal{A}(x)} \mathbb{P}(A^{x,\pi})$ by (16), we conclude.
2. Now, we show the reverse inequality $V_1(x) \leq \widetilde{V}(x)$. Let $A \in \mathcal{F}_T$ be an arbitrary set satisfying the constraint $\sup_{Z\in\mathcal{Z}} \mathbb{E}[ZFI_A] \leq x$. This implies a super-replication by some $\pi \in \mathcal{A}(x)$ such that $\mathbb{P}\{X^{x,\pi}(T) \geq FI_A\} = 1$. In turns, this yields $\mathbb{P}\{X^{x,\pi} \geq F\} \geq \mathbb{P}(A)$. Therefore, $\widetilde{V}(x) \geq \mathbb{P}(A)$ by (16). Thanks to the arbitrariness of A, $\widetilde{V}(x) \geq V_1(x)$ holds.

In conclusion, $\widetilde{V}(x) = V_1(x)$, and the strategy π super-hedging FI_A is the solution of (16), provided that $\widetilde{V}(x) = \mathbb{P}(A)$.

Next, we give an example where we explicitly compute the maximum success probability. By Proposition 1, this is equivalent to the pure hypothesis testing, but not to the randomized analogue, as shown below.

Example 2. Consider $\Omega = \{0, 1\}$, $\mathcal{F} = 2^{\{0,1\}}$, and the real probability given by $\mathbb{P}\{0\} = \mathbb{P}\{1\} = 1/2$. Suppose stock price $S(t, \omega)$ follows one-step binomial tree:

$$S(0,0) = S(0,1) = 2; \quad S(T,0) = 5, \; S(T,1) = 1.$$

The benchmark $F = 1$ at T. We will determine by direct computation the maximum success probability given initial capital $x \geq 0$. To this end, we notice that the possible strategy with initial capital x is c shares of stock plus $x - 2c$ dollars of cash at $t = 0$. Then, the terminal wealth $X(T)$ is

$$X(T) = \begin{cases} 5c + (x - 2c) = x + 3c & \omega = 0, \\ c + (x - 2c) = x - c & \omega = 1. \end{cases}$$

Since $X(T) \geq 0$ a.s. is required, we have constraint on c, i.e.$-\frac{x}{3} \leq c \leq x$. Now, we can write $\widetilde{V}(x)$ as

$$\widetilde{V}(x) = \max_{-\frac{x}{3} \leq c \leq x} \mathbb{P}\{X(T) \geq 1\} = \frac{1}{2} \max_{-\frac{x}{3} \leq c \leq x} \Big(I_{\{x+3c\geq 1\}} + I_{\{x-c\geq 1\}} \Big). \tag{18}$$

As a result, we have

1. If $x < 1/4$, then $x + 3c \leq x + 3x = 4x < 1$ and $x - c \leq x + \frac{x}{3} = \frac{4x}{3} < 1/3$, which implies both indicators are zero, i.e. $\widetilde{V}(x) = 0$.
2. If $1/4 \leq x < 1$, then we can take $c = 1/4$, which leads to $x + 3c \geq 1$, i.e. $\widetilde{V}(x) \geq 1/2$. On the other hand, $\widetilde{V}(x) < 1$. From this and (18), we conclude that $\widetilde{V}(x) = 1/2$.
3. If $x \geq 1$, then we can take $c = 0$, and $\widetilde{V}(x) = 1$.

Hence, we conclude that $\widetilde{V}(x) = V_1(x) \neq V(x)$ of Example 1.

Theorem 4. *Suppose the super-hedging price $F_0 < \infty$, and one of two conditions are satisfied*

1. *$\mathcal{Z}$ is a singleton, and $\mathcal{F}_T$ contains a random variable with continuous cumulative distribution w.r.t. $\mathbb{P}$;*
2. *$\mathbb{P}\{ZF = 1\} = 0$ for all $(a, Z) \in (0, \infty) \times \mathcal{Z}$.*

Then,

(i) *$V(x)$ of (16) is continuous, concave, and non-decresing in $x \in [0, \infty)$, taking values from the minimum $V(0) = \mathbb{P}\{F = 0\}$ to the maximum $V(x) = 1$ for $x \geq F_0$.*
(ii) *There exists $\hat{A} \in \mathcal{F}_T$ that solves (17), and*

$$V(x) = \inf_{a \geq 0, Z \in \mathcal{Z}} \mathbb{E}[xa + (1 - aZF)^+]. \tag{19}$$

Proof. This is an application of Theorems 2 and 3 where we take $\mathcal{H} = \{FZ : Z \in \mathcal{Z}\}$ and $\mathcal{G} = \{1\}$. Note that $F_0 < \infty$ implies $\mathcal{H}$ is L^1 bounded. Under this setup, Assumption 1 is satisfied.

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