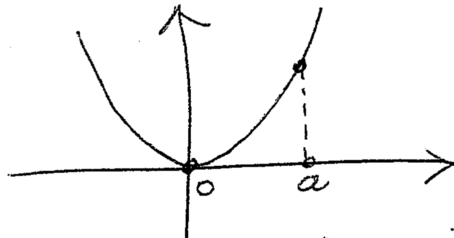
Arclength for a Parabola



$$L[0,a] = \int_0^a \sqrt{1+(f'(x))^2}\, dx$$

$$y = f(x) = x^2 \qquad = \int_0^a \sqrt{1+4x^2}\, dx$$

$$f'(x) = 2x$$

Let $x = \frac{1}{2}\tan(\theta)$. Then

$$dx = d\theta/(2\cos^2(\theta))$$

$$\sqrt{1+4x^2} = \sqrt{1+\tan^2(\theta)}$$

$$= \sqrt{\frac{\cos^2(\theta)+\sin^2(\theta)}{\cos^2(\theta)}}$$

$$\sqrt{1+4x^2} = 1/\cos(\theta)$$

$$\therefore L[0,a] = \frac{1}{2}\int_{x=0}^{x=a} \frac{d\theta}{\cos^3(\theta)}, \quad \underline{\theta = \arctan(2x)}$$

$$\therefore L[0,a] = \frac{1}{2}\int_0^{\arctan(2a)} \frac{d\theta}{\cos^3(\theta)}$$

Fact: $\int \frac{d\theta}{\cos^3(\theta)} = \frac{1}{2}\left[\frac{\tan(\theta)}{\cos(\theta)} + \ln\left|\frac{(1+\sin(\theta))}{\cos(\theta)}\right|\right] + k$

$$\Rightarrow L[0,a] = \frac{1}{4}\left[\frac{\tan(\theta)}{\cos(\theta)} + \ln\left|\frac{(1+\sin(\theta))}{\cos(\theta)}\right|\right]$$

where $\theta = \arctan(2a)$.

In the case $a=1$,

$$\theta = \arctan(2)$$

$$\frac{\sin(\theta)}{\cos(\theta)} = 2$$

$$\Rightarrow \sin(\theta) = 2\cos(\theta).$$

Let $s = \sin(\theta)$,
$c = \cos(\theta)$,
$t = \tan(\theta) = s/c = 2.$

Then $s^2 + c^2 = 1$ & $s = 2c$

$$\Rightarrow 4c^2 + c^2 = 1$$

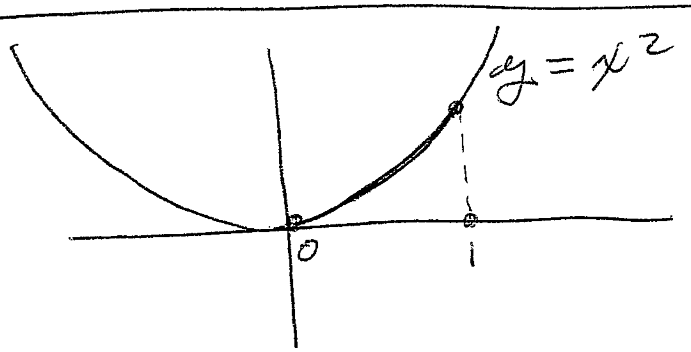
$$\Rightarrow 5c^2 = 1 \Rightarrow \underline{c = 1/\sqrt{5}}.$$

& $s = 2c = 2/\sqrt{5}.$

$$L[0,1] = \frac{1}{4}\left[\frac{t}{c} + \ln\left(\frac{1+s}{c}\right)\right]$$

$$= \frac{1}{4}\left[2\sqrt{5} + \ln\left(\left(1+\frac{2}{\sqrt{5}}\right)\sqrt{5}\right)\right]$$

$$\boxed{L[0,1] = \frac{\sqrt{5}}{2} + \frac{1}{4}\ln(\sqrt{5}+2)}$$



$$\int c^{-3}(d\theta) = \int c^{-1}\,dt = c^{-1}t - \int t\,dc^{-1}$$

$$= c^{-1}t - \int t\,c^{-2}s$$

$$= c^{-1}t - \int s^{2}c^{-3}$$

$$= c^{-1}t - \int (1-c^{2})\,c^{-3}$$

$$= c^{-1}t - \int c^{-3} + \int c^{-1}$$

$$\Rightarrow \quad \int c^{-3} = \frac{1}{2}\left[c^{-1}t + \int c^{-1}\right]$$

$$d\left((1+s)c^{-1}\right) = c\,c^{-1} + (1+s)\,c^{-2}s$$

$$= (c^{2}+s+s^{2})\,c^{-2}$$

$$= \left((1+s)c^{-1}\right)c^{-1}$$

$$\Rightarrow \quad c^{-1} = d\left((1+s)c^{-1}\right)/\left((1+s)c^{-1}\right)$$

$$\Rightarrow \quad \int c^{-1} = \ln\left|(1+s)c^{-1}\right| + k$$

$$\Rightarrow \quad \boxed{\int c^{-3} = \frac{1}{2}\left[c^{-1}t + \ln\left|(1+s)c^{-1}\right|\right] + k}$$