Calculus Notes

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I. Questions

The most commonly asked question in a mathematics course is:

Will this be on the exam?

Well not everything will be on the exam, and so this is certainly a legitimate question.

On the other hand, there are lots of things that will not be on the exam that work in your favor if you understand them and can work with them. The calculus course is based on a few very simple and powerful ideas.

In order to make use of ideas and make them work for you it is necessary to think and recall and comprehend.
You need to use curiosity and imagination. You need to be willing to look for meaning and logic. You need to appreciate structure.
You need to be willing to calculate and work and ask questions.

The question with which we began this section is not sufficient!
In a very real sense, you need to question everything.

Why does mathematics work the way it does?
What is the nature of number and geometry? How does mathematics manage to model complex phenomena in the world? What is the internal structure of mathematics? How do questions in mathematics become interesting? What sort of mathematical problems do you find easy? Which ones are hard? What is mathematics? Is it calculation, reasoning, problem solving, a language, an art form, a mental discipline, a form of philosophical investigation?

What are the largest mathematical ideas that you know?
What is a number?
What is the meaning of continuity?
What is the nature of the discrete and the continuous?
What beliefs are in back of mathematics as we do it?
What beliefs are in back of the physical modeling that is so

tightly related to mathematics?
What is the nature of time and space?
How does mathematics have anything to do with time and space?

We want you to ask questions.
We want you to ask questions whose answers really mean a lot to you. We want you to ask questions that have nothing to do with the exams and everything to do with understanding.

What should I do in order to ace this course?
Ask questions.
Exercise curiosity, attention, intelligence and imagination.
Work hard.
Use logic.
Think about the material in the course.
Teach the material in the course!
Teach it to another student.
Teach it to yourself.
Teach it to the professor by asking him really tough questions about how calculus works and how it is applied!

II. Integration and Differentiation

Calculus is based on a small number of key ideas. The purpose of this section is to give you a quick introduction to these ideas that will be useful as a start and as a reference as we progress in the course.

Three Big Ideas

Calculus is all about three basic mathematical notions.
1. Deriviatives = DIFFERENCES
and
2. Integrals = SUMMATIONS
and
3. LIMITS.
Please come back to this remark after you have read the rest of the section!

Certainly differences and sums are related to one another.
For example,
$(2-1)+(3-2) +(4-3) + (5-4) + (6-5) + ... +(99 - 98) + (100-99)$
$= 100 -1 = 99$.
A sum of differences can be a bigger difference because all the little differences cancel each other out.

Taking differences can reveal a rule.
What is the rule for the following sequence?

1,6,11,16,21,26,31,36,41,...

Limits are interesting to think about. For example, you may be able to see at once that
1 + 1/2 + 1/4 + 1/8 + 1/16 + 1/32 + 1/64 + ...
has limiting value equal to 2 as the number of terms in this sum becomes arbitrarily large. What is the geometric interpretation of this claim?
Another example:

$$\pi^2/6 = 1/1 + 1/4 + 1/9 + 1/16 + 1/25 + 1/36 + 1/49 + \ldots$$

$\pi^2/6$ is equal to the sum of the reciprocals of all the squares.
This last example is a deeper result than the first series. Try checking it on your calculator.

Differentiation
Consider the following problems:
1. You drive a car for one hour and cover a distance of 60 miles. What is your average velocity?
Answer: 60 miles/hour.

2. In driving a car for one hour you cover **21** miles in the first **15** minutes. How fast should you drive in the remaining **45** minutes in order to cover a total of **60** miles in the hour? What is your average velocity for the whole hour?
Answer: You need to cover **39** miles in **45** minutes. That is a velocity of **39** miles/(**3/4** hour) = (**4/3**)**39**miles/hour = **52** miles/hour.
Your velocity in the first **15** minutes was
21miles/(**1/4** hour) = **84** miles/hour. (You were speeding!)
Your average velocity for the hour was
(**1/4**)(**84** mph) + (**3/4**)(**52** mph) = (**84** + **156**)/**4** mph
= **240/4** mph = **60** mph.

Notice that this average velocity is the same as the total number of miles covered divided by the total time. Can you figure out why it worked out this way? I will leave the question of why it works in general to class discussion, but the arithmetic of the situation reveals a pattern:

AvgVelocity $= (1/4)(84) + (3/4)(52)$

$=(1/4)(21/(1/4)) + (3/4)(60 - 21)/(3/4))$

$=21 + (60-21)$

$=60.$

Discuss what happened here. Will it work in other problems about average velocity?

3. In driving a car you start off by accelerating your car. The distance you travel while you are accelerating is given by the formula $X(t) = t^2$ where t is in minutes and $X(t)$ is in miles. You only accelerate in this way for two minutes. How far do you travel in one minute? If you kept on accelerating, how far would you go in 10 minutes?

What is your velocity at time $t = 1$ minute?

Answer: $X(1) = 1$, so you travel one mile in one minute.

$X(10) = 100$ miles, so you would go 100 miles in ten minutes.

Now to see approximately how fast your are going at one minute, lets first see how far you go in a time Δt after 1 minute. We have

$X(1) = 1$ and

$X(1 + \Delta t) = (1 + \Delta t)^2 = 1 + 2\Delta t + (\Delta t)^2$.

This means that the distance you travel in time Δt, after already traveling one mile is

$\Delta X = X(1 + \Delta t) - X(1) = 1 + 2\Delta t + (\Delta t)^2 - 1 = 2\Delta t + (\Delta t)^2.$

So your average velocity in the time interval Δt is

$\Delta X/\Delta t = (2\Delta t + (\Delta t)^2)/\Delta t = 2 + \Delta t$ miles per minute.

This is $120 + 60\Delta t$ miles per hour.

Your velocity is varying as a function of time, but if we make the time interval Δt very short, we can estimate your "instantaneous velocity" at time $t = 1$ as the limiting value of $120 + 60\Delta t$ as Δt goes to zero. This limiting value is clearly **120**. So we can say that you are going at **120** miles per hour after one minute.

4. You would be crazy to keep on accelerating at the rate given in the previous problem. But lets calculate some other limiting velocities. Given that your distance function is $X(t) = t^2$, what is your instantaneous velocity at an arbitrary positive time t?

Answer:

$\Delta X(t) = X(t + \Delta t) - X(t) = (t + \Delta t)^2 - t^2$

$= (t^2 + 2t\Delta t + (\Delta t)^2) - t^2$

$= 2t\Delta t + (\Delta t)^2.$

Thus

$\Delta X(t)/\Delta t = 2t + \Delta t.$

We conclude from this that the instantaneous velocity of our car at time **t** is **2t** miles per minute, which is equal to **120t** miles per hour where **t** is measured in minutes. At the **30** second point you are going **60** miles per hour. If you keep accelerating at this rate, you will be going **240** miles per hour after two minutes and **1200** miles per hour after **10** minutes!

The method by which we have solved these problems is called *differentiation*. The general pattern is just the same as in our example for velocity. We have a function $F(t)$ (it was $X(t) = t^2$ in our example) and we define the difference quotient by the formula

$$\Delta F(t)/\Delta t = (F(t + \Delta t) - F(t))/\Delta t$$

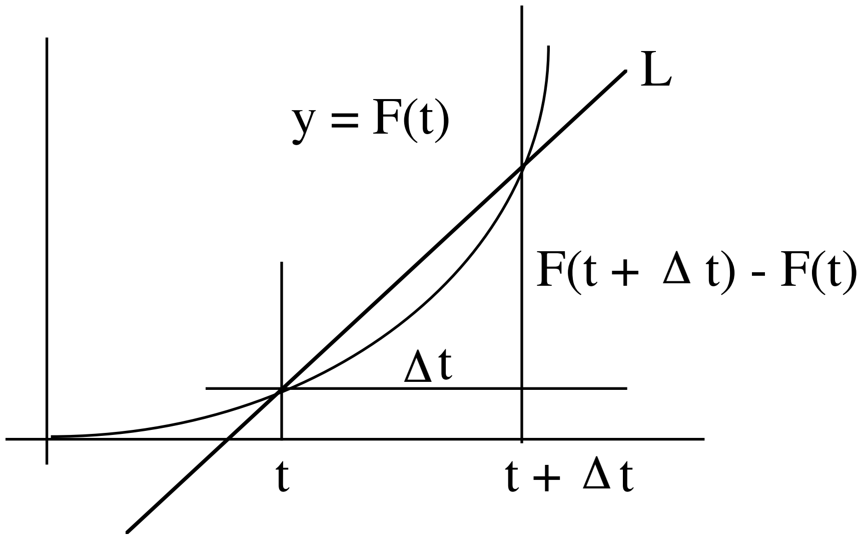
The *derivative* of $F(t)$ at t is defined by taking the *limit* of $\Delta F/\Delta t$ as Δt goes to zero. We denote this limit by $dF(t)/dt$ and sometimes by $F'(t)$. Thus we write

$$d(t^2)/dt = 2t$$

via the calculations that we did in the problem solving above.

Tangent Lines and Derivatives

There is a beautiful interpretation of the difference quotient as an approximation to the slope of the tangent line to the graph of a function, and the derivative as the actual slope of the tangent line to the graph at a given point.



The main aspect of this interpretation is illustrated in the figure above. Note that the line **L** intersecting the graph of **y = F(t)** has slope **(F(t + Δt) - F(t))/Δt = ΔF(t)/Δt**. As **Δt** approaches zero the line **L** approaches the tangent line to the graph at the point **(t, F(t))** and the difference quotient approaches **dF(t)/dt.** Thus the derivative of the function **F(t)** at **t** is the slope of the tangent line to its graph at that point.

Discussion

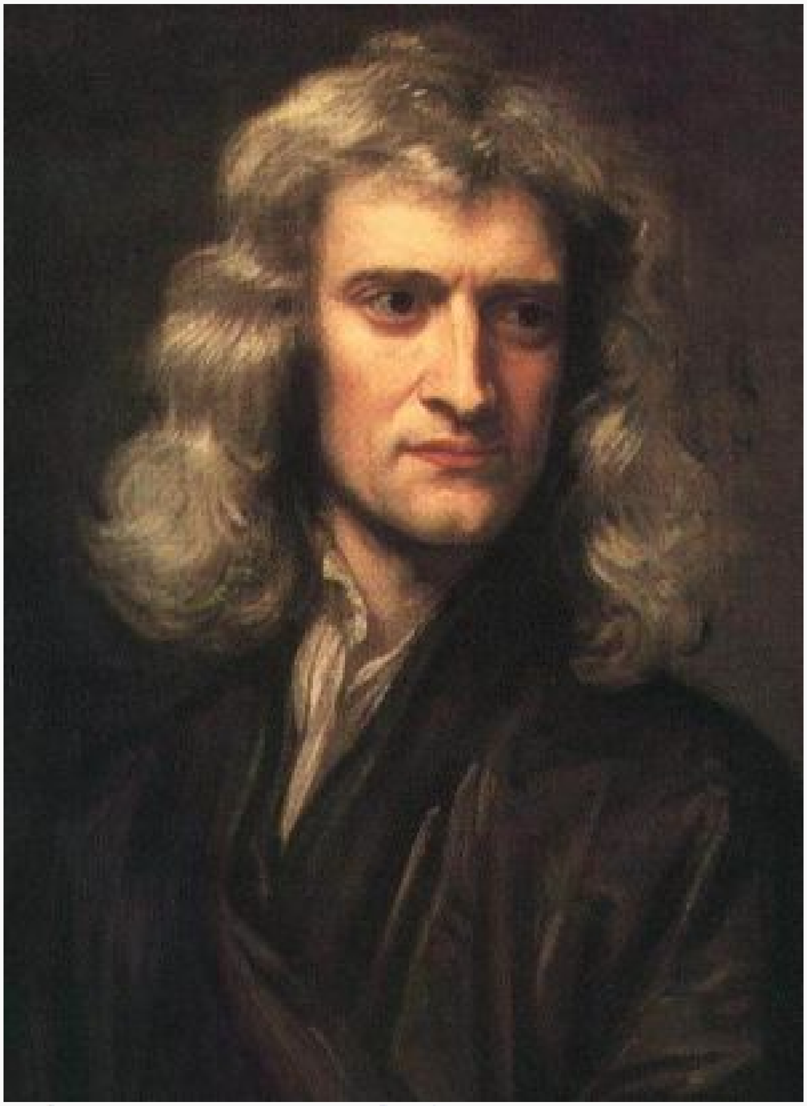
As you can see, the study of calculus will involve a combination of algebra, geometry and the art of taking limits. In this way studying calculus involves all the mathematics that you have learned up to this point.

Calculus was invented/discovered over 200 years ago by Isaac Newton <http://en.wikipedia.org/wiki/Isaac_Newton> and independently by Gottfried Wilhelm Leibniz <http://mally.stanford.edu/leibniz.html>.

Gottfried Wilhelm Leibniz

(b. 1646, d. 1716) was a German philosopher, mathematician, and logician who wrote about mathematics, logic, science, history, law and theology.)

Isaac Newton



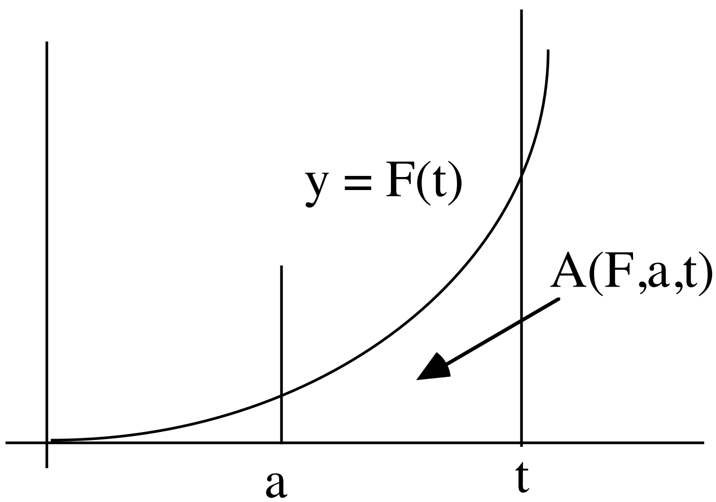
Sir Isaac Newton at 46 in Godfrey Kneller's 1689 portrait

Born	4 January 1643 [OS: 25 December 1642][1] Woolsthorpe-by-Colsterworth, Lincolnshire, England
Died	31 March 1727 [OS: 20 March 1727][1] Kensington, London, England
Occupation	Physicist, mathematician, astronomer, alchemist, and natural philosopher

Newton discovered the calculus as an aid for doing dynamical physics to describe gravity and the motions of the planets. Leibniz discovered the calculus in the course of mathematical and philosophical investigations. The notation **dF/dt** is due to Leibniz. Leibniz dreamed of a universal calculus that could be used to settle all problems involving reasoning. His dream is unrealized to this day, but the evolution of calculus, mathematical logic, computer programming and computer science is, from our modern point of view, the beginning of a possible realization of that dream.

Integration

One of the great results of the calculus (due to Newton and Leibniz) is its application to the problem of finding areas and volumes. Consider the following problem: Find the area between the graph of a function **y = F(t)** and the **t**-axis between the values **a** and **t**. Lets call this area **A(t) = A(F, a, t)**.



Note that the area **A(t)** under the curve from **a** to **t** is a function of **t**. We can ask what is the rate of change of this area as a function of **t**? In other words, what is **dA(t)/dt**?
The remarkable answer is the

THE FUNDAMENTAL THEOREM OF CALCULUS:

$$\mathbf{dA(F,a,t)/dt = F(t).}$$

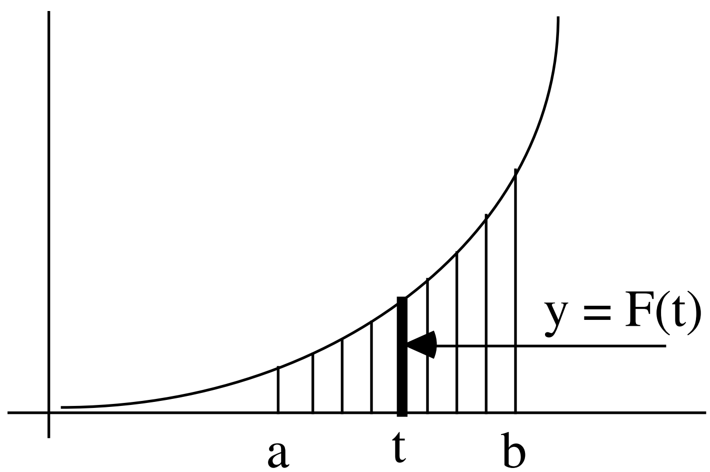
Because we shall develop lots of techniques for differentiation we will be able to use this theorem to calculate all sorts of areas. The art of finding areas using calculus is called *integral calculus*.

It is not hard to see pictorially why the fundamental theorem of calculus is true. Consider the difference

$$\Delta A = (A(t + \Delta t) - A(t)).$$

This is the area of a small strip between **t** and **t + Δt,** and is well-approximated by **(base x height) = Δt F(t)**. Therefore the difference quotient **ΔA/Δt** is approximately equal to **F(t)**, and this approximation gets better and better as **Δt** goes to zero.

Integrals are Limits of Sums.
One way to think about the area under a curve is to cut the area up into many vertical strips and take the sum of the areas of the strips.



The base of a given strip has length **Δt**. If **Δt** is very small, then the area of the strip will be approximately **F(t)Δt**. Thus the integral of **y = F(t)** from **t=a** to **t=b** is approximately the sum of the numbers **F(t)Δt** as t ranges over a finite but ever-larger subdivision of the interval from **a** to **b**. We have illustrated such a subdivision in the figure above.

Notation for the Integral
The Leibniz notation for the area **A(F,a,b)** (the area under the graph **y = f(t)** between **t =** a and **t** = **b**) is

$$\mathbf{A(F,a,b)} = \int_a^b F(t)\,dt.$$

The notation is designed to remind us that the area can be approximated by dividing it into many small vertical strips with height **F(t)** and base **Δt.** We than take the limit of a such a finite sum to get the actual area. The Leibniz notation for the integral reminds us of this by replacing Δt by **dt** and the summation by the long **S**, $\int$ that represents the limit of the sum.

We can then restate the fundamental theorem of calculus with the formula

$$\mathbf{d}\left(\int_a^x F(t)dt\right)/\mathbf{dx} \;=\; \mathbf{d/dx}\left(\int_a^x F(t)dt\right) \;=\; \mathbf{F(x).}$$

There is much more to be said about the relationship between differentiation and integration. We will be covering much of this as the course goes along. For example, we would like to find the area from **t = a to t = b** under the parabola $\mathbf{y = t^2}$. This is represented by the integral $\int_a^b \mathbf{t^2dt}$**.** The fundamental theorem of calculus tells us

$$\mathbf{d/dx}\left(\int_a^x \mathbf{t^2dt}\right) = \mathbf{x^2.}$$

It also turns out that $\mathbf{d(x^3)/dx = 3x^2}$**.**
You can work this derivative out for yourself in the same way that we worked out the derivative of $\mathbf{x^2}$**.** (Note that we have changed from **t** to **x** here.) This implies that $\mathbf{d(x^3/3)/dx = x^2}$. So the function $\mathbf{x^3/3}$ has the same derivative as the area function for the parabola. A little more work shows that one has the area formula

$$\int_a^b \mathbf{t^2dt = b^3/3 - a^3/3.}$$

This is an example of how calculus can solve area problems that involve curved and complex boundaries.

Discussion

We have now covered the main ideas. These ideas are the key to learning differential and integral calculus.

Advanced Problems

1. Recall our early problem about average velocity. Now suppose that we know that the positions of our car at the times
$t_0=a,\ t_1, t_2, ..., t_{n-1}, t_n = b$ are
$X_0, X_1, X_2, ..., X_{n-1}, X_n$ respectively.

The total distance travelled is $X = X_n - X_0$.
With this notation, we can think about lots of examples.
We will take the velocity of the car in the time interval
$[t_k, t_{k+1}] = \{t \mid t_k < t < t_{k+1}\}$ to be
$V_k = (X_{k+1} - X_k)/(t_{k+1} - t_k)$.
We let $\Delta k = t_{k+1} - t_k$. Here k runs from 0 to $n-1$.

Since the car has velocity V_k in the time interval Δk, the average velocity of the car over the time interval $[a,b]$ is equal to

AvgVelocity =
$V_0\ \Delta_k/T + V_1\ \Delta_k/T + V_2\ \Delta_k/T + ... + V_{n-1}\ \Delta_k/T$

where $\Delta_k/T = (t_{k+1} - t_k)/(b-a)$ is the fraction of the time the car spent at the velocity V_k.

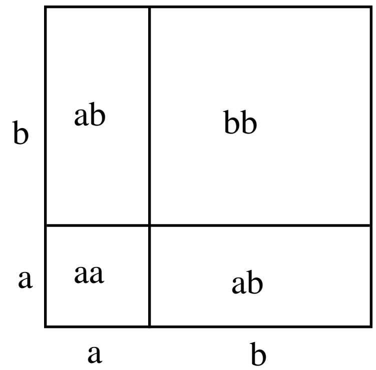
(a) Work out an example of your choice for this problem using $n = 4$, with numbers for the times and the positions. Check that the average velocity does work out to be equal to X/T where X is the total distance travelled, and T is the total time for the journey.

(b) Show in general, by using algebra, that
the average velocity is equal to X/T where X is the total distance travelled, and T is the total time for the journey.
where X is the total distance travelled and T is the total time for the trip.

2. (a) Convince yourself by multiplying out the algebra that

$$(a + b)^3 = a^3 + b^3 + 3a^2b + 3ab^2.$$

Given that we have the two-dimensional geometric interpretation of $(a+b)^2 = a^2 + b^2 + 2ab$ as shown below,



find a *three-dimensional* geometric interpretation for the formula for $(\mathbf{a} + \mathbf{b})^3$.

(b) Use the results of part (a) of this problem to show that $\mathbf{d(t^3)/dt = 3t^2}$.

(c) Generalize everything and show that $\mathbf{d(t^n)/dt = (n\text{-}1)t^{n-1}}$.

III. Background

1. Basic arithmetic.

This includes how to use numbers: integers (positive and negative), fractions, decimals and the notion of real numbers as possibly inifinite decimal expansions. Here are a few problems to check your knowledge.

(a) Find the sum of the first 100 positive integers, starting with 1 and ending with 100.
$1 + 2 + 3 + ... + 98 + 99 + 100 = ?$

(b) Consider the sums
$1 = 1$
$1 + 3 = 4$
$1 + 3 + 5 = 9$
$1 + 3 + 5 + 7 = 16$
What is the pattern? If you can see the pattern for the sum of the odd numbers from 1 to 2n+1, can you find a way to convince yourself that the pattern will always work?

(c) What is the value of $2/3 + 7/11$ as a fraction of the form p/q?

(d) Why is the following equality true:
$1/3 = .33333...$ (with infinitely many 3's in this decimal expansion).

(e) Find the complete decimal expansion for 1/7. No calculuators allowed.

(f) What is the value of
1 + 1/2 + 1/4 + 1/8 + 1/16 + 1/32 + ... ?

(g) Here is a method for checking a sum. First we define the "digital root" of a number as follows: Add the digits of the number to get a new number. Then add the digits of that number and keep on going until you reach a single digit. That digit is the digital root of the original number.
For example: Start with 123775.
Add: 1 + 2 + 3 + 7 + 7 + 5 = 25
Add: 2 + 5 = 7.
The digital root of 123775 is 7.
To use digital roots to check sums, find the digital roots of each number in the sum. Add the digital roots together and take the digital root of that. The answer should be the same as the digital root of the sum of the original numbers. For example:

```
     123 -----------> 6
     245 -----------> 2
   + 776 -----------> 2
-----------           ----
   1144   --------->  1
```

The arrows point to the digital roots of the numbers. Note that 1 (which is the digital root of 1144) is the digital root of 6+2+2.
Try some examples of this method.
Why does it work?

(h) Use long division to show that 235/17 = 13 + 14/17.
Can you explain why the long-division method works?

(i) What is the largest number that you can name?
Is there a largest number?
What does it mean when we say that there are infinitely many numbers?

(j) Explain why you get a repeating decimal expansion when you convert a fraction to a decimal. (e.g. 1/3 = .3333...) We call fractions "rational numbers". Show that there are infinitely many real numbers (as infinite decimals) that are not rational.

(hint: consider examples like
0.1010010001000010000010000001...)

(k) Why do we say that $1 = 0.999999....$?

(l) I claim that the square root of two is not rational:

Let $S^2 = 2$. (So S is the square root of two.)
Suppose that $S = p/q$ for some positive integers p and q. We can assume that p and q are not both even numbers by assuming that this is a reduced fraction. (We would replace 14/6 by 7/3.) Then
$S=p/q$ implies
$Sq = p$ and this implies that
$S^2q^2 = p^2$ so that
$2q^2 = p^2$.
This last equation tells us that p2 is even. Since the square of on odd number is odd, we conclude that p must be even. Suppose that $p = 2n$. then $p^2 = 4n^2$. So
$2q^2 = 4n^2$ and hence
$q^2 = 2n^2$.
But this shows that q is even. This is a contradiction since we know that not both p and q are even. The contradiction tells us that the the square root of two is not rational.

Explain this argument to a friend. What does the irrationality of the square root of two say about its decimal expansion?

2. Basic Algebra, Geometry and Graphing
(a) Recall that $(a+b)^2 = a^2 + 2ab + b^2$. Give a geometric interpretation for this formula.

(b) What is the formula for $(a+b)^3$? Can you make a three dimensional interpretation of this formula?

(c) $Ax^2 + Bx + C = 0$. Tell everything you know about solving this quadratic equation without looking in a book. In particular, suppose that $x + y = 3$ and $xy = 4$. Find all solutions x and y to this equation. (Hint: Eliminate y and rewrite x as a solution to a quadratic equation.)

(d) Draw sketches of the graphs of

$$y = x^2 + 1,$$
$$y = x^2 - 1,$$
$$y = 1/x,$$
$$y = (x-1)(x-2)(x-3).$$

(e) Check that the following identity is true

$$(x^2-y^2)^2 + (2xy)^2 = (x^2+y^2)^2.$$

What does this algebra identity have to do with the Pythagorean Theorem?

(f) Suppose that $p = a^3+b^3$ and $q = 3ab$ and $x = a+b$.
Show that $x^3 = p + qx$.

(g) Suppose that you are given p and q, and you are told that $p = a^3+b^3$ and $q = 3ab$. Find the values of a and b. Use these results to find the roots of the cubic equation $x^3 = 1+ 3x$.

3. Basic Geometry and Trigonometry

(a) State the Pythagorean Theorem. (People often answer this by saying $a^2 + b^2 = c^2$. Note that you need to explain what are a, b and c that they should be related in this way.)

(b) Why is the sum of the angles of triangle equal to 180 degrees?

(c) Explain the construction, using ruler and compass for bisecting an angle.

(d) Define the sine and cosine functions with reference to right triangles. Find sine and cosine of the angles 30, 60 and 45 degrees.

(e) Let **i** denote the square root of minus one so that $\mathbf{i}^2 = -1$.
Why can't **i** be an ordinary real number?
What is the geometric interpretation of the square root of minus one?

(f) Recall that

$$\mathbf{\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y),}$$
$$\mathbf{\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y),}$$
$$\mathbf{\sin^2(x) + \cos^2(x) = 1.}$$

What is the simple geometric interpretation of this last formula?

(g) Recall that complex numbers are built from real numbers by using the square root of minus one . A typical complex number is **3 + 4i**. We multiply complex numbers like this:

$$\mathbf{(3 + 4i)(1 + 7i) =}$$
$$\mathbf{3(1+7i) + 4i(1 +7i) =}$$
$$\mathbf{3 + 21i + 4i + 28i^2 =}$$
$$\mathbf{3 + 21i + 4i - 28 =}$$
$$\mathbf{-25 + 25i.}$$

More generally we have

$$\mathbf{(a + bi)(c + di) = (ac -bd)+ (ad + bc)i.}$$

Work out the product
(cos(x) + i sin(x))(cos(y) + i sin(y)) and simplify it. Discuss your answer from the point of view of the geometric interpretation of the square root of minus one.

(f) Recall that π is defined to be the ratio of the circumference of a circle to its diameter. Thus we have the formula

$$\mathbf{C = 2\pi r}$$

relating the circumference and the radius of the circle. (The diameter is twice the radius.)
The area of a circle is given by the formula

$$\mathbf{A = \pi r^2}$$

where r is the radius of the circle. How closely can you measure π by an experiment?

(g) The Grand International Railroad Company has decided to encircle the earth's equator with a circular railway track, running like a belt around the earth. They have 30 feet more track than the circumference of the earth. Thus their railroad belt around the earth will, in principle, have a larger radius than the earth itself. How high above the ground must the Company construct their track? (Use an idealized earth of constant radius and no obstructions.)