Model Theory 1: Homework 2

Due on Friday September 29th

1. Let $\mathcal{L}$ be a finite language and $\mathcal{A}$ a finite $\mathcal{L}$-structure.
   a) Show that there is a single $\mathcal{L}$-sentence $\phi_{\mathcal{A}}$ such that, for any $\mathcal{L}$-structure $\mathcal{B}$,
   $$\mathcal{A} \cong \mathcal{B} \iff \mathcal{B} \models \phi_{\mathcal{A}}.$$
   b) Show that, for any such sentence $\phi_{\mathcal{A}}$, the theory $\{\phi_{\mathcal{A}}\}$ is complete.

2. Let $L = \{s\}$ where s is a unary function symbol. Let $T = \mathrm{Th}(\mathbb{N}, \mathrm{succ})$, where succ is the successor function. For what cardinals κ is T κ-categorical?
   [Hint: You should give a description of all models of T.]

3. Suppose that $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$ is a tower of $\mathcal{L}$-structures such that $\mathcal{A} \preccurlyeq \mathcal{C}$ and $\mathcal{B} \preccurlyeq \mathcal{C}$. Show that $\mathcal{A} \preccurlyeq \mathcal{B}$.

4. An exercise in non-standard analysis.
   Let $\mathcal{R}$ be a proper elementary extension of the real field $\langle \mathbb{R}; <, +, \cdot, 0, 1\rangle$. Elements of $\mathbb{R}$ are known as *standard reals*.
   a) Show that there is an infinitesimal element of $\mathcal{R}$, that is, a nonzero α such that for every positive standard real a, $-a < \alpha < a$.
   b) Show that the sum of two infinitesimals is infinitesimal or zero; the product of two infinitesimals is infinitesimal; and the product of an infinitesimal with a nonzero standard real is infinitesimal.
   c) An easy way to compute the derivative of a polynomial:
   Let $f \in \mathbb{R}[X]$ be a polynomial with standard real coefficients. Show that there is a unique polynomial $g \in \mathbb{R}[X]$ such that for any standard real x and any infinitesimal α, there is a β which is infinitesimal or zero such that
   $$\frac{f(x+\alpha) - f(x)}{\alpha} = g(x) + \beta.$$
   [Hint: First take f to be a monomial X^n, then look at g for $f_1 + f_2$ and for rf where r is a standard real.]