Some basic facts about vector spaces

Fix a field F, and take V and W to be F-vector spaces. (For definitions, look at any linear algebra book, algebra book, or online.) Some of the notions of linear algebra are usually introduced only for finite dimensional vector spaces. This note is intended to give some basic properties for infinite dimensional spaces as well.

A subset X of V is *linearly independent* iff whenever $x_1, \ldots, x_n \in X$ and $\lambda_1, \ldots, \lambda_n \in F$ are such that $\sum_{i=1}^n \lambda_i x_i = 0$ then each $\lambda_i = 0$.

A subset X of V is a *spanning set* for V iff every element of V can be written as a (finite) linear combination of elements of X. (Note that there is no notion of an infinite linear combination here – that would require some topology and a notion of convergence.)

A *basis* of V is a subset B of V such that every element $v \in V$ can be uniquely written as a sum $v = \sum_{b \in B_0} \lambda_b b$ for some finite subset B_0 of B. Equivalently, a basis is a linearly independent spanning set.

Two important facts are as follows:

Lemma 1. *Every linearly independent set can be extended to a basis.*

(The proof of this requires Zorn's lemma.)

Lemma 2. *Any two bases of V have the same cardinality.*

This allows us to define the dimension of a vector space to be the cardinality of a basis.

A homomorphism of vector spaces is called a *linear map*. The easiest way to define a linear map from V to W is as follows. Let B be a basis of V. Let f_0 be any function from B to W. Then f_0 extends uniquely to a linear map $f : V \to W$ by defining $f\left(\sum_{b \in B_0} \lambda_b b\right) = \sum_{b \in B_0} \lambda_b f_0(b)$. By the definition of a basis this is a well-defined function on V. It is easy to see that it is a linear map.

Let B' be the image of B. Note that this map f is injective precisely when B' is linearly independent and f_0 is injective. Note that f is surjective precisely when B' is a spanning set of W. Thus f is a bijection precisely when f_0 is a bijection between bases of V and W. Also note that a bijective linear map is invertible, so a bijective linear map is an isomorphism of vector spaces. In particular, two F-vector spaces are isomorphic precisely when they have the same dimension.