Maximally symmetric trees

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December 1, 2000

To John Stallings on his 64.8^{th} *birthday*

Abstract

We characterize the "best" model geometries for the class of virtually free groups, and we show that there is a countable infinity of distinct "best" model geometries in an appropriate sense–these are the maximally symmetric trees. The first theorem gives several equivalent conditions on a bounded valence, cocompact tree T without valence 1 vertices saying that T is maximally symmetric. The second theorem gives general constructions for maximally symmetric trees, showing for instance that every virtually free group has a maximally symmetric tree for a model geometry.

1 Introduction

A *model geometry* for a finitely generated group G is a proper metric space X on which G acts properly and coboundedly by isometries; equivalently, there is a discrete, cocompact, finite kernel representation $G \to \operatorname{Isom} X$. The group G with its word metric is quasi-isometric to each of its (coarse) geodesic model geometries (see §2.3).

Given a quasi-isometry class $\mathcal{C}$ of finitely generated groups, one can ask:

(1) Is there a common (coarse) geodesic model geometry X for every group in $\mathcal{C}$?

(2) Is there a common locally compact group Γ, in which every group of $\mathcal{C}$ has a discrete, cocompact, finite kernel representation?

A "yes" to (1) implies a "yes" to (2), with or without the word "coarse". We show in Corollary 7 that the coarse version of (1) is equivalent to (2).

Many quasi-isometric rigidity theorems take the form of a positive answer to these questions. For example, if X is an irreducible nonpositively curved symmetric space then X is itself a model geometry for every group quasi-isometric to X. For $X = \mathbf{H}^n$ see [Sul81], [Tuk86], [CC92], and for $\mathbf{CH}^n$ see [KR95], [Cho96]; in these cases, every cobounded quasi-action on X is quasiconjugate to an isometric action on X. For $\mathbf{QH}^n$ and the Cayley hyperbolic plane see [Pan89], and for X of higher rank see [KL97] and also [EF97]; in these cases, every quasi-isometry is a bounded distance from an isometry.

By contrast, consider the class of groups $\mathcal{VF}$ which are virtually free of finite rank ≥ 2. $\mathcal{VF}$ is a single quasi-isometry class, as follows from Stallings ends theorem [Sta68], [SW79] and Dunwoody's accessibility theorem [Dun85]. $\mathcal{VF}$ coincides with the class of

fundamental groups of finite graphs of finite groups [KPS73]. The typical model geometries for $\mathcal{VF}$ are bounded valence, bushy trees; any quasi-action on such a tree T is quasiconjugate to an isometric action on a *possibly different* tree T' ([MSW00]; see Theorem 8 below). For the class $\mathcal{VF}$, questions (1), (2) have a negative answer [MSW00]: for primes $p \neq q \geq 3$ the groups $\mathbf{Z}/p * \mathbf{Z}/p$ and $\mathbf{Z}/q * \mathbf{Z}/q$ have *no* discrete, cocompact, virtually faithful representations in the same locally compact group.

In lieu of a single model geometry for $\mathcal{VF}$, we describe "best" model geometries X, namely those which are "maximally symmetric". Roughly speaking this means that any continuous, proper, cocompact embedding of $\operatorname{Isom} X$ into another locally compact group is an isomorphism. This is asking for too much, though: one can always take the product of X with a symmetric compact metric space; or equivariantly attach to each point of some discrete orbit of X a symmetric pointed compact metric space. We should therefore avoid compact normal subgroups, and for $\mathcal{VF}$ this is acheived by restricting to bounded valence, bushy trees with no valence 1 vertices. Our main results show how to recognize maximally symmetric trees within this class, in quasi-isometric, topological, and graph theoretical terms, and how to construct maximally symmetric trees wherever needed; for example, each group in $\mathcal{VF}$ has some maximally symmetric tree as a model geometry.

Statements of results Quasi-isometries are reviewed in §2.2. For any metric space X the *quasi-isometry group* $\mathrm{QI}(X)$ is the group of self quasi-isometries of X modulo identification of quasi-isometries which have bounded distance in the sup norm. Any quasi-isometry $f\colon X \to Y$ induces an isomorphism $\mathrm{ad}_f\colon \mathrm{QI}(X) \to \mathrm{QI}(Y)$, and so any finitely generated group with the word metric has the same quasi-isometry group as any of its model geometries.

Let X be a δ-hyperbolic for some $\delta \geq 0$. A subgroup $H < \mathrm{QI}(X)$ is *uniform* if its elements can be represented by quasi-isometries of X with uniform quasi-isometry constants. Equivalently, H can be represented by a quasi-action on X; moreover, any two such quasi-actions differ in the sup norm by a bounded amount. We say that H is *cobounded* if a representing quasi-action is cobounded.

A tree T has *bounded valence* if each vertex has valence $\leq C$ for some constant C, and T is *bushy* if each vertex is a uniformly bounded distance from some vertex v such that at least three components of $T - v$ are unbounded. Any two bounded valence, bushy trees are quasi-isometric, and so their quasi-isometry groups are isomorphic. A bounded valence, bushy tree T is *cocompact* if $\operatorname{Isom} T$ acts cocompactly on T, or equivalently if the image of the natural homomorphism $\operatorname{Isom} T \to \mathrm{QI}(T)$ is cobounded. A *thorn* of T is a valence 1 vertex; if T is thornless then $\operatorname{Isom} T$ has no compact normal subgroups, and the homomorphism $\operatorname{Isom} T \to \mathrm{QI}(T)$ is injective, among other nice properties.

Theorem 1 (Characterizing maximally symmetric trees). *For any bounded valence, bushy, cocompact, thornless tree T, the following are equivalent:*

- $\operatorname{Isom} T$ *is a maximal uniform cobounded subgroup of* $\mathrm{QI}(T)$.
- *For any bounded valence, bushy, thornless tree T', any continuous, proper, cocompact embedding* $\operatorname{Isom} T \to \operatorname{Isom} T'$ *is an isomorphism.*
- *For any locally compact group $\mathcal{G}$ without compact normal subgroups, any continuous, proper, cocompact embedding* $\operatorname{Isom} T \to \mathcal{G}$ *is an isomorphism.*

Such trees T are called *maximally symmetric*.

The proof of Theorem 1 is entirely abstract and nonconstructive: it does not exhibit the existence of a single maximally symmetric tree, let alone showing that they are model geometries for virtually free groups. These facts follow from Theorem 2 whose proof is concrete and constructive; as a byproduct, we obtain a finitistic method for enumerating the isometry types of maximally symmetric trees.

A bounded valence, bushy, cocompact tree T is said to be *index 1 normalized* if, for every vertex v and every incident edge e, the stabilizers of v and e in $\mathrm{Isom}\, T$ satisfy $[\mathrm{Stab}(v) : \mathrm{Stab}(e)] \geq 2$. Note that index 1 normalized implies thornless.

Theorem 2 (Existence of maximally symmetric trees). *Fix a bounded valence, bushy tree τ.*

(1) Every uniform cobounded subgroup of $\mathrm{QI}(\tau)$ is contained in a maximal uniform cobounded subgroup.

(2) For every maximal uniform cobounded subgroup $\mathcal{G} < \mathrm{QI}(\tau)$ there exists a maximally symmetric tree T which is index 1 normalized, and there exists a quasi-isometry $f\colon T \to \tau$, such that $\mathcal{G} = \mathrm{ad}_f(\mathrm{Isom}\, T)$. Moreover T and f are uniquely specified in the following sense: if $\mathcal{G} = \mathrm{ad}_{f'}(\mathrm{Isom}\, T')$ for another index 1 normalized T' and quasi-isometry $f'\colon T' \to \tau$, then there exists an isometry $h\colon T \to T'$ such that $\mathrm{ad}_f = \mathrm{ad}_{f'} \circ \mathrm{ad}_h$.

(3) There is a natural one-to-one correspondence between conjugacy classes of maximal uniform cobounded subgroups of $\mathrm{QI}(\tau)$ and isometry classes of index 1 normalized, maximally symmetric trees T.

(4) There is a countable infinity of such isometry classes; in fact there is a countable infinity of both unimodular and nonunimodular isometry classes.

Part (2) can be interpreted as saying that *any* maximally symmetric model geometry in the quasi-isometry class of τ is, in a certain sense, equivalent to a maximally symmetric *tree*. In part (4), unimodularity of a tree T means that the locally compact group $\mathrm{Isom}\, T$ is unimodular, i.e. each left invariant Haar measure on $\mathrm{Isom}\, T$ is also right invariant. Unimodularity was shown by Bass and Kulkarni [BK90] to be equivalent to the existence of a discrete, cocompact subgroup of $\mathrm{Isom}\, T$.

Corollary 3. *For every group $G \in \mathcal{VF}$ there exists a maximally symmetric tree T which is a model geometry for G, and so G is the fundamental group of a finite graph of finite groups $\Gamma = T/G$ whose Bass-Serre tree T is maximally symmetric.*

More information about Corollary 3 can be extracted from the proof of Theorem 2, namely an algorithm which inputs any finite graph of finite groups, and outputs another one with the same fundamental group whose Bass-Serre tree is maximally symmetric.

Our results should be compared and contrasted with results concerning an irreducible, nonpositively curved symmetric space X. For instance, recent results of Alex Furman [Fur00] show that X is "maximally symmetric" in a manner very similar to that described in Theorem 1. Contrasting with our Theorem 2, Furman's results can be interpreted as saying that X is the unique best model geometry in its quasi-isometry class; also, the

quasi-isometric rigidity theorems quoted above show that $\mathrm{QI}(X)$ has a unique maximal uniform cobounded subgroup up to conjugacy, namely $\mathrm{Isom}\, X$.

The proofs of Theorems 1 and 2 use the rigidity theorem for quasi-actions on trees from [MSW00], the theory of edge-indexed graphs from [Bas93] and [BK90], and tree techniques reminiscent of [BL94]. In particular, one the main technical steps is Proposition 22, which gives conditions on the quotient graphs of bounded valence, bushy, thornless trees T, T' that are sufficient to prove that any continuous, proper, cocompact monomorphism $\mathrm{Isom}\, T \to \mathrm{Isom}\, T'$ is an isomorphism induced by an isometry $T \to T'$. Proposition 22 is related to a result of Bass and Lubotzky ([BL94], Corollary 4.8(d)), which obtains the same conclusion under somewhat stronger conditions.

The proof of Theorem 2 gives a finitistic characterization of maximally symmetric trees in terms of edge-indexed graphs; see Corollary 24. For example, the bihomogeneous tree $T_{p,q}$ of alternating valences $p > q \geq 2$ is maximally symmetric; the quotient edge-indexed graph $T_{p,q}/\,\mathrm{Isom}\, T_{p,q}$ has two vertices and one edge, with one end of index p and the other end of index q. In general, to each bounded valence, bushy tree T there corresponds a quotient edge-indexed graph $T/\,\mathrm{Isom}\, T$; conversely to each edge-indexed graph Γ there corresponds a "universal covering tree" T and a deck transformation group $D(\Gamma) < \mathrm{Isom}\, T$. With respect to this correspondence, Corollary 24 describes a certain subclass of edge-indexed graphs Γ whose isomorphism classes are in one-to-one correspondence with maximally symmetric trees. Also, Corollary 21 gives a simple algorithm which, given an edge indexed graph Γ, decides whether Γ belongs to this subclass, and if not then the algorithm computes another edge-indexed graph Γ' which does belong, and for which there is an nonsurjective embedding $D(\Gamma) < D(\Gamma')$ which is continuous, proper, and cocompact. This algorithm can be used to prove Corollary 3, starting from a finite graph of groups Γ with fundamental group G.

In the unimodular case, part (4) of Theorem 2 can be summarized by saying that there is a countable infinity of "best" geometries for the class $\mathcal{VF}$. Part (4) is proved by simply giving some examples, but we will improve that by showing in Proposition 25 that any finite graph Γ in which no edge is a loop and no two edges have the same endpoints has infinitely many distinct edge-indexings, both unimodular and (when Γ is not a tree) nonunimodular, corresponding to a maximally symmetric tree. The proof of Proposition 25 will give an effective, one-to-one enumeration of the isometry classes of maximally symmetric trees.

While our focus in this introduction is mostly on the unimodular case, Theorems 1 and 2 apply also to nonunimodular trees. This may be applicable to graphs of groups having bounded valence, bushy, Bass-Serre trees, in situations where these trees can be nonunimodular, such as graphs of $\mathbf{Z}$'s, graphs of $\mathbf{Z}^n$'s, etc.

Acknowledgements

The authors are supported in part by the National Science Foundation: the first author by NSF grant DMS-9803396; the second author by NSF grant DMS-989032; and the third author by an NSF Postdoctoral Research Fellowship.

2 Preliminaries

2.1 Metric spaces

A metric space X is *proper* if closed balls are compact. This implies that X is complete, and that the isometry group $\operatorname{Isom} X$ is locally compact and Hausdorff in the compact open topology. An action of a group G on X will always mean an isometric action, that is, a homomorphism $\phi\colon G \to \operatorname{Isom} X$, usually written $g \mapsto \phi_g \in \operatorname{Isom} X$. Properness of X implies that an action is cocompact if and only if it is cobounded, if and only if image(ϕ) is a cocompact subgroup of $\operatorname{Isom} X$ (a co-P action is one for which there is a P-subset K the union of whose translates equals the whole space). The action ϕ is *properly discontinuous* if for any two compact sets $K, L \subset X$ the set $\{g \in G \mid \phi_g(K) \cap L \neq \emptyset\}$ is finite. When X is proper, an action ϕ is properly discontinuous if and only if image(ϕ) is a discrete subgroup of $\operatorname{Isom} X$ and $\operatorname{Ker}(\phi)$ is a finite subgroup of G.

2.2 Quasi-isometries

A *quasi-isometry* between two metric spaces X, Y is a map $f\colon X \to Y$ such that for some constants $K \geq 1$, $C \geq 0$ we have

$$\frac{1}{K} d_X(x,y) - C \leq d_Y(fx, fy) \leq K d_X(x,y) + C, \quad x, y \in X$$

and for all $y \in Y$ there exists $x \in X$ such that $d_Y(fx, y) \leq C$. Every K, C quasi-isometry $f\colon X \to Y$ has a *coarse inverse*, which is a K, C' quasi-isometry $\bar{f}\colon Y \to X$ such that $d_{\sup}(\bar{f} \circ f, \mathrm{Id}_X) \leq C'$ and $d_{\sup}(f \circ \bar{f}, \mathrm{Id}_Y) \leq C'$, where the constant C' depends only on K, C; the notation $d_{\sup}$ denotes the sup metric on functions.

The *quasi-isometry group* of a metric space X, denoted $\mathrm{QI}(X)$, is defined as follows. Let $\widehat{\mathrm{QI}}(X)$ denote the set of quasi-isometries, equipped with the operation of composition. Define $f, g \in \widehat{\mathrm{QI}}(X)$ to be *coarsely equivalent* if $d_{\sup}(f, g) < \infty$, and let $[f]$ be the coarse equivalence class. Note that composition is well-defined on coarse equivalence classes, thereby making the set of coarse equivalence classes into a group $\mathrm{QI}(X)$. The inverse of the coarse equivalence class of $f \in \widehat{\mathrm{QI}}(X)$ is the class of any coarse inverse for f.

Given a quasi-isometry $f\colon X \to Y$ there is an induced isomorphism $\mathrm{ad}_f\colon \mathrm{QI}(X) \to \mathrm{QI}(Y)$ defined by $\mathrm{ad}_f[g] = [f \circ g \circ \bar{f}]$ for any coarse inverse $\bar{f}$ of f.

A *quasi-action* of a group G on a metric space X is a map $A\colon G \to \widehat{\mathrm{QI}}(X)$, denoted $g \mapsto A_g$, such that for some $K \geq 1$, $C \geq 0$ we have: each map A_g is a K, C quasi-isometry; $d_{\sup}(A_{\mathrm{Id}}, \mathrm{Id}) < C$; and for each $g, g' \in G$ we have $d_{\sup}(A_g \circ A_{g'}, A_{gg'}) < C$. Postcomposing the quasi-action $G \xrightarrow{A} \widehat{\mathrm{QI}}(X)$ with the quotient map $\widehat{\mathrm{QI}}(X) \to \mathrm{QI}(X)$ we obtain the induced homomorphism $G \to \mathrm{QI}(X)$. The quasi-action A is *cobounded* if there exists a bounded subset $D \subset X$ such that for each $x \in X$ there is a $g \in G$ with $A_g(x) \in D$; also, A is *proper* if for each $R > 0$ there exists an integer $M > 0$ such that for each $x, y \in X$ the cardinality of the set $\{g \in G \mid d(A_g(x), y) \leq R\}$ is at most M. Given quasi-actions A, B of G on metric spaces X, Y respectively, a *quasiconjugacy* from A to B is a quasi-isometry $f\colon X \to Y$ such that for some $C \geq 0$ we have $d_{\sup}(f \circ A_g, B_g \circ f) \leq C$, for all $g \in G$; it follows that $\mathrm{ad}_f[A_g] = [B_g]$. Coboundedness and properness are quasiconjugacy invariants of quasi-actions.

For quasi-isometries among hyperbolic metric spaces, boundary values coarsely determine a quasi-isometry, in the following sense. For any δ, K, C there exists A such that if $f, g\colon X \to Y$ are K, C quasi-isometries between proper, geodesic, δ-hyperbolic metric spaces X, Y, and if the boundary extensions $\partial f, \partial g\colon \partial X \to \partial Y$ are identical, then $d_{\sup}(f, g) \leq A$. It follows that when X is δ-hyperbolic, the following two properties on a subgroup $H < \mathrm{QI}(X)$ are equivalent: H is *uniform*, meaning that each element of H is represented by a K, C quasi-isometry for some fixed $K \geq 1$, $C \geq 0$; H has an *induced quasi-action*, namely a quasi-action $s\colon H \to \widehat{\mathrm{QI}}(X)$ such that the composition $H \xrightarrow{s} \widehat{\mathrm{QI}}(X) \to \mathrm{QI}(X)$ equals the inclusion. When H is uniform, any two induced quasi-actions $s, s'\colon H \to \widehat{\mathrm{QI}}(X)$ differ by a bounded distance in the sup norm, that is, $\sup\{d_{\sup}(sh, s'h) \mid h \in H\} < \infty$. A uniform subgroup $H < \mathrm{QI}(X)$ is *cobounded* if some (and hence any) induced quasi-action of H on X is cobounded.

2.3 Coarse geodesic metric spaces

Sections 2.3 and 2.4 contain the proof of Corollary 7, that the coarse version of question (1) in the introduction is equivalent to question (2). Beyond the basic definitions, most of the material of these two subsections will not be needed for the rest of the paper.

In a metric space X, a *geodesic* joining x to y is a path $\alpha\colon [a, b] \to X$ such that $x = \alpha(a)$, $y = \alpha(b)$, and $d(\alpha(s), \alpha(t)) = |s - t|$ for $s, t \in [a, b]$. We say that X is a *geodesic metric space* if any two points are joined by a geodesic.

A *coarse path* joining x to y is just a sequence $x = x_0, \ldots, x_n = y$ in X; the *word length* equals n, and the *path length* equals $\sum_{i=1}^{n} d(x_{i-1}, x_i)$. We say that $x_0, \ldots, x_n$ is a *C-coarse path* if $d(x_{i-1}, x_i) \leq C$ for $i = 1, \ldots, n$. A *C-coarse geodesic* is a C-coarse path whose path length equals the distance between its endpoints. A metric space X is a *coarse geodesic metric space* if there exists $C \geq 0$ such that any two points are joined by a C-coarse geodesic.

A proper metric space X is geodesic if and only if $d(x, y)$ is the infimum of the path lengths of all rectifiable paths joining x to y. The next lemma, applied to the collection $\mathcal{V}$ of closed balls of radius C, shows similarly that a proper metric space X is C-coarse geodesic if and only if $d(x, y)$ is the infimum of the path lengths of C-coarse geodesics joining x and y; we need a more general version of this fact for later purposes.

We generalize the notion of a C-coarse path as follows. Let $\mathcal{V} = \{V(x) \mid x \in X\}$ where for each x the set $V(x) \subset X$ is a compact neighborhood of x, and the following symmetry condition holds: $x \in V(y)$ if and only if $y \in V(x)$. A *$\mathcal{V}$-coarse path* is a coarse path $x_0, \ldots, x_n$ such that $x_i \in V(x_{i-1})$ for $i = 1, \ldots, n$. We say that X is *$\mathcal{V}$-coarsely connected* if any two points $x, y \in X$ can be joined by a $\mathcal{V}$-coarse path. In this case we define the *$\mathcal{V}$-word metric* $\mu_{\mathcal{V}}(x, y)$ to be the shortest word length of a $\mathcal{V}$-coarse path joining x to y, and the *$\mathcal{V}$-path metric* $\rho_{\mathcal{V}}(x, y)$ to be the infimum of the path lengths of all $\mathcal{V}$-coarse paths joining x to y.

Lemma 4. *Let X be a proper metric space, and suppose that $\mathcal{V} = \{V(x)\}$ is as above, and that $\mathcal{V}$ satisfies the following:*

- *X is $\mathcal{V}$-coarsely connected.*
- *There exists $R > r > 0$ such that for each $x \in X$,*

$$\overline{B}(x, r) \subset V(x) \subset \overline{B}(x, R)$$

Then: $\rho_{\mathcal{V}}$ is a coarse geodesic metric whose restriction to each ball of radius r agrees with the given metric on X; and the metrics $\mu_{\mathcal{V}}$ and $\rho_{\mathcal{V}}$ are quasi-isometric, that is, the identity map is a quasi-isometry between $\mu_{\mathcal{V}}$ and $\rho_{\mathcal{V}}$.

Proof. The idea of the proof is that $\rho_{\mathcal{V}}$ is a "maximal metric" in the sense of Gromov [Gro93], subject to the constraint that $\rho_{\mathcal{V}}$ agrees locally with the given metric on X.

Given $x, y \in X$ and $n \geq \mu_{\mathcal{V}}(x, y)$, define $\|x, y\|_n$ to be the infimum of the path lengths of all $\mathcal{V}$-coarse paths joining x to y which have word length $\leq n$. Note that the sequence $\|x, y\|_n$ is nonincreasing and has limit $\rho_{\mathcal{V}}(x, y)$.

Fix n for the moment. Since X is proper, the infimum defining $\|x, y\|_n$ is acheived by some $\mathcal{V}$-coarse path $x = x_0, \ldots, x_k = y$ with a minimal word length $k = k(n)$, $\mu_{\mathcal{V}}(x, y) \leq k \leq n$. Note that the path $x_0, \ldots, x_k$ cannot have a subpath x_{i-1}, x_i, x_{i+1} such that each of $d(x_{i-1}, x_i)$, $d(x_i, x_{i+1})$ is $\leq r/2$ because then $d(x_{i-1}, x_{i+1}) \leq r$ which would produce a $\mathcal{V}$-coarse path $x = x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k$ of path length $\leq \|x, y\|_n$ whose word length is smaller than $k(n)$, a contradiction. It follows that at least $\lfloor k(n)/2 \rfloor$ of the distances $d(x_{i-1}, x_i)$ are $> r/2$ (where $\lfloor \bullet \rfloor$ denotes the greatest integer function). We therefore have

$$\|x, y\|_n > (k(n) - 1)C/4$$

Now the sequence $k(n)$ is evidently nondecreasing; moreover, $\|x, y\|_{n+1} < \|x, y\|_n$ if and only if $k(n) < k(n + 1) = n + 1$. If $k(n)$ is not bounded above it follows that $\|x, y\|_n$ diverges to $+\infty$, a contradiction. Therefore $k(n)$ is eventually constant, proving that $\|x, y\|_n$ is eventually constant and equal to $\rho_{\mathcal{V}}(x, y)$. This shows that $\rho_{\mathcal{V}}(x, y)$ is a coarse geodesic metric.

Now we compare $\rho_{\mathcal{V}}$ to $\mu_{\mathcal{V}}$. Obviously

$$\rho_{\mathcal{V}}(x, y) \leq R \cdot \mu_{\mathcal{V}}(x, y)$$

For the other direction, we have seen that $\rho_{\mathcal{V}}(x, y)$ is realized by some $\mathcal{V}$-coarse path of least word length k, and the argument shows that

$$\mu_{\mathcal{V}}(x, y) \leq k < \frac{4}{C}\rho_{\mathcal{V}}(x, y) + 1$$

$\diamondsuit$

2.4 Locally compact, compactly generated groups

All locally compact groups are assumed to be Hausdorff. For example, from the Ascoli-Arzela theorem it follows that the isometry group of a proper metric space is locally compact Hausdorff, in the compact open topology.

The next lemma says that locally compact, compactly generated groups, like finitely generated groups, have a well-defined geometry up to quasi-isometry. Moreover, just as finite index implies quasi-isometry among finitely generated groups, "compact index" implies quasi-isometry among compactly generated groups.

Lemma 5. *Let $\mathcal{G}$ be a locally compact topological group, G a closed, cocompact subgroup. Then $\mathcal{G}$ is compactly generated if and only if G is compactly generated. Moreover, if this is so then the inclusion $G \to \mathcal{G}$ is a quasi-isometry with respect to the compactly generated word metrics. Finally, any two compactly generated word metrics on $\mathcal{G}$ are quasi-isometric.*

Proof. If we substitute "finitely generated" for "compactly generated", and "finite index" for "cocompact", then this is a standard result, and the proof goes through unchanged, with one caveat. To show that two finite generating sets A, B determine quasi-isometric word metrics one must prove that $A \subset B^n$ and $B \subset A^m$ for some integers n, m. We must prove the same when A, B are compact generating sets.

First we reduce to the case of compact generating sets containing a neighborhood of the identity e. Supposing that A is any compact generating set, it follows that $\cup_{i=1}^{\infty} A^n = \mathcal{G}$, and so by the Baire category theorem some A^i contains an open ball B. Also, some A^j contains e. Therefore, A^{i+j} contains a neighborhood of e, and we can replace A by A^{i+j}.

Letting A, B be two compact generating sets each containing a neighborhood of e, for each $x \in B$ there exists i such that A^i contains a neighborhood of x, and by compactness of B it follows that $B \subset A^m$ for some m; similarly $A \subset B^n$. $\diamondsuit$

Remark on the proof Note that $\mathbf{R}$ has a compact generating set with empty interior. Namely, if E is any compact set with empty interior and positive measure, then the set $E - E = \{e_1 - e_2 \mid e_1, e_2 \in E\}$ contains a neighborhood of 0, by an application of the Lebesgue density theorem, and so $E \cup -E$ generates $\mathbf{R}$ and has empty interior.

The disadvantage of Lemma 5 is that a compactly generated word metric does *not* determine the correct topology on $\mathcal{G}$: indeed, if the generating set contains a neighborhood of the origin then the word metric is discrete. We correct this, at the same time obtaining a coarse geodesic metric, as follows:

Lemma 6. *Suppose that $\mathcal{G}$ is a locally compact, compactly generated group. Then there exists a left invariant coarse geodesic metric ρ on $\mathcal{G}$ such that ρ yields the given topology on $\mathcal{G}$ and ρ is quasi-isometric to any compactly generated word metric on $\mathcal{G}$.*

Proof. By a result of Birkhoff [Bir36] and of Kakutani [Kak36], we know that there exists a left invariant metric D on $\mathcal{G}$ yielding the topology on $\mathcal{G}$. Let V be a compact generating set for $\mathcal{G}$; by enlarging V we may assume that V contains the D-ball of some radius $r > 0$ about e, and that $V = V^{-1}$. Let $\mathcal{V} = \{g \cdot V \mid g \in G\}$. Applying Lemma 4, the $\mathcal{V}$-coarse geodesic metric $\rho_{\mathcal{V}}$ agrees with D on each D-ball of radius $< r$, and $\rho_{\mathcal{V}}$ is quasi-isometric to the compactly generated word metric $\mu_{\mathcal{V}}$. Moreover, $\rho_{\mathcal{V}}$ is clearly left invariant. $\diamondsuit$

It follows that any group G with a discrete, cocompact, finite kernel representation to $\mathcal{G}$ preserves ρ under the left action of G on $\mathcal{G}$, and so regarding $\mathcal{G}$ as a coarse geodesic,

proper metric space X, we obtain a properly discontinuous, cocompact action of G on X. This proves:

Corollary 7. *Given a collection of groups $\mathcal{C}$, the following are equivalent:*

(1) There exists a coarse geodesic metric space X on which each group in $\mathcal{C}$ acts properly discontinuously and cocompactly.

(2) There exists a locally compact group $\mathcal{G}$ in which each group of $\mathcal{C}$ has a discrete, cocompact, finite kernel representation.

$\diamondsuit$

2.5 Graphs

A *graph* Γ is a connected 1-complex, and a *tree* is a contractible graph. A *vertex* of Γ means a 0-cell; the set of vertices is denoted $\mathrm{Verts}(\Gamma)$. An *edge* means a 1-cell, that is, a component of the complement of the vertices; the set of edges is denoted $\mathrm{Edges}(\Gamma)$. For each edge e we choose a compact arc $\overline{e}$ and a characteristic map $(\overline{e}, \partial\overline{e}) \to (\Gamma, \mathrm{Verts}(\Gamma))$ taking $\mathrm{int}(\overline{e})$ homemorphically to e. Each edge e has two *ends* in the sense of Freudenthal, corresponding one-to-one with the endpoints of $\overline{e}$, this correspondence being denoted $\eta \leftrightarrow p_\eta$; the set of ends of e is denoted $\mathrm{Ends}(e)$. Each $\eta \in \mathrm{Ends}(e)$ is *located* at a particular vertex of Γ, namely the unique limit point in Γ of the end η, identified with the image of p_η under the characteristic map. Denote $\mathrm{Ends}(\Gamma) = \cup_{e \in \mathrm{Edges}(\Gamma)} \mathrm{Ends}(e)$. The set of $e \in \mathrm{Ends}(\Gamma)$ located at a particular vertex $v \in \mathrm{Verts}(\Gamma)$ is denoted $\mathrm{Ends}(v)$; this corresponds to the "link" of v. We denote $\mathrm{Ends}(e, v) = \mathrm{Ends}(e) \cap \mathrm{Ends}(v)$, the set of ends of e located at v, a set of cardinality zero, one, or two. An edge e is called a *loop* if there exists $v \in \mathrm{Verts}(\Gamma)$ such that $\mathrm{Ends}(e, v) = \mathrm{Ends}(e)$. Note that we do not adopt a preferred orientation for an edge, the distinction between the two orientations being encoded in the two ends.

We impose on each graph Γ a geodesic metric in which edge has length 1. The *isometry group* $\mathrm{Isom}\,\Gamma$ is defined to be the group of cellular isometries of Γ; this coincides with the usual isometry group except in the single case when Γ is isometric to the real line. The group $\mathrm{Isom}\,\Gamma$, with the compact-open topology, is locally compact Hausdorff. Moreover, if $\mathrm{Isom}\,\Gamma$ acts cocompactly on Γ then $\mathrm{Isom}\,\Gamma$ is compactly generated: letting Δ be any finite subgraph of Γ whose translates under $\mathrm{Isom}\,\Gamma$ cover Γ, the set $K_\Delta = \{g \in \mathrm{Isom}\,\Gamma \mid g(\Delta) \cap \Delta \neq \emptyset\}$ is a compact generating set.

3 Characterizing maximally symmetric trees

Given a bounded valence, bushy, thornless, cocompact tree T, to prove Theorem 1 we must prove the equivalence of the following properties, which we may then take as the definition of *maximally symmetric*:

(1) For any locally compact group $\mathcal{G}$ with no compact normal subgroups, any continuous, proper, cocompact monomorphism $\mathrm{Isom}\,T \to \mathcal{G}$ is an isomorphism.

(2) For any bounded valence, bushy, thornless tree T', any continuous, proper, cocompact monomorphism $\mathrm{Isom}\, T \to \mathrm{Isom}\, T'$ is an isomorphism.

(3) $\mathrm{Isom}\, T$ is a maximal uniform cobounded subgroup of $\mathrm{QI}(T)$.

Proof that (1) implies (2). Obvious. ♢

Proof that (2) implies (3). Suppose that $\mathrm{Isom}\, T < A$ for some uniform cobounded subgroup A of $\mathrm{QI}(T)$. Choose an induced cobounded quasi-action $s\colon A \to \widehat{\mathrm{QI}}(T)$; as remarked at the end of §2.2, s is unique up to bounded distance in the sup norm. Now we apply the main result of [MSW00]:

Theorem 8 (Rigidity of quasi-actions on trees). *If T is a bounded valence, bushy tree and $s\colon G \to \widehat{\mathrm{QI}}(T)$ is a quasi-action of a group G on T, then there exists an action $s'\colon G \to \mathrm{Isom}\, T'$ of G on a bounded valence, bushy tree T', and there exists a quasiconjugacy $f\colon T \to T'$ from s to s'.* ♢

We obtain a quasiconjugacy $f\colon T \to T'$ from the quasi-action $s\colon A \to \widehat{\mathrm{QI}}(T)$ to an injective cobounded action $s'\colon A \to \mathrm{Isom}\, T'$. Restricting to $\mathrm{Isom}\, T$ gives an injective action $s'\colon \mathrm{Isom}\, T \to \mathrm{Isom}\, T'$ which is quasiconjugate via f to the canonical action of $\mathrm{Isom}\, T$ on T. Since $\mathrm{Isom}\, T$ is cobounded on T it follows that $s'(\mathrm{Isom}\, T) < \mathrm{Isom}\, T'$ is cobounded, that is, cocompact, on T'.

Now we need a lemma from [MSW00]:

Lemma 9. *Given a bounded valence, bushy tree T, a sequence (g_i) converges in* $\mathrm{Isom}\, T$ *if and only if (g_i) satisfies the following property:*

Coarse convergence *There is a number D so that for any v there is an n so that the set $\{g_i(v) \mid i \geq n\}$ has diameter at most D.*

♢

A convergent sequence $g_i \in \mathrm{Isom}\, T$ clearly satisfies coarse convergence. Since coarse convergence is clearly invariant under quasiconjugacy, the image sequence $s'(g_i) \in \mathrm{Isom}\, T'$ also satisfies coarse convergence. Applying Lemma 9 it follows that $s'(g_i)$ converges in $\mathrm{Isom}\, T'$, proving that $s'\colon \mathrm{Isom}\, T \to \mathrm{Isom}\, T'$ is continuous. Also, s' is proper, for suppose $C \subset \mathrm{Isom}\, T'$ is compact. Choose a sequence $g_i \in s'^{-1}(C)$. Passing to a subsequence, $s'(g_i)$ converges to some $h \in C$. It follows that $s'(g_i)$ satisfies coarse convergence in T', and again by quasiconjugacy invariance it follows that g_i satisfies coarse convergence in T. Applying Lemma 9 it follows that g_i converges in $\mathrm{Isom}\, T$ to some g. By continuity of s' we have $s'(g) = h$ and so $g \in s'^{-1}(C)$, proving that $s'^{-1}(C)$ is compact and so s' is proper.

Having proved that s' is continuous, proper, and cocompact, applying (2) it follows that s' is surjective, which implies that $\mathrm{Isom}\, T = A$, proving that $\mathrm{Isom}\, T$ is maximal in $\mathrm{QI}(T)$. ♢

Proof that (3) implies (1). Assuming (3) is true suppose that we have an embedding $\iota\colon \mathrm{Isom}\, T \to \mathcal{G}$ as in (1).

Let Δ be a compact fundamental domain for T and consider the compact generating set $K_\Delta = \{f \in \mathrm{Isom}\, T \mid f(\Delta) \cap \Delta \neq \emptyset\}$ for $\mathrm{Isom}\, T$. The left-invariant word metric on

$\operatorname{Isom} T$ determined by the generating set K_Δ is quasi-isometric to the tree T. Specifically, the map $F\colon T \to \operatorname{Isom} T$, taking a vertex $w \in T$ to any isometry $F_w \in \operatorname{Isom} T$ such that $w \in F_w(K_\Delta)$, is a quasi-isometry from T to $\operatorname{Isom} T$.

We have a quasi-isometry $F\colon T \to \operatorname{Isom} T$, and applying Lemma 5 the injection $\iota\colon \operatorname{Isom} T \to \mathcal{G}$ is a quasi-isometry. The left action of $\mathcal{G}$ on itself is clearly a cobounded quasi-action, and quasiconjugating via $\iota \circ F\colon T \to \mathcal{G}$ we obtain a cobounded quasi-action of $\mathcal{G}$ on T. Applying Theorem 8 produces a quasiconjugacy $\Phi\colon T \to T'$ from the $\mathcal{G}$ quasi-action on T to a cobounded action $A\colon \mathcal{G} \to \operatorname{Isom} T'$ for some bounded valence, bushy tree T'.

Repeating the argument above using Lemma 9, the homomorphism $\mathcal{G} \xrightarrow{A} \operatorname{Isom} T'$ is continuous, proper, and cocompact. Properness implies that the kernel is compact, but the group $\mathcal{G}$ having no compact normal subgroup, it follows that $\mathcal{G} \xrightarrow{A} \operatorname{Isom} T'$ is an embedding. Letting $\bar\Phi\colon T' \to T$ be a coarse inverse of Φ, this shows that $\operatorname{Isom} T < \operatorname{ad}_{\bar\Phi}(\operatorname{Isom} T')$, and the latter is clearly a uniform, cobounded subgroup of $\mathrm{QI}(T)$. Applying (3) it follows that $\operatorname{Isom} T = \operatorname{ad}_{\bar\Phi}(\operatorname{Isom} T')$, which implies that the composition of injections $\operatorname{Isom} T \xrightarrow{\iota} \mathcal{G} \xrightarrow{A} \operatorname{Isom} T'$ is an isomorphism, and so $\operatorname{Isom} T \xrightarrow{\iota} \mathcal{G}$ is surjective. $\diamondsuit$

4 Edge indexed graphs

In this section we show how edge-indexed graphs can be used to encode bounded valence trees. The material on graphs of groups and edge-indexed graphs is taken for the most part from [Bas93] and [BK90].

4.1 Graphs of groups

For detailed references see [Ser80], [Bas93], and [SW79] for the more topological viewpoint. We adopt a different notation for graphs than these references.

A *graph of groups* is a graph Γ together with a *vertex group* Γ_v for each $v \in \mathrm{Verts}(\Gamma)$, an *edge group* Γ_e for each $e \in \mathrm{Edges}(\Gamma)$, and an *edge-to-vertex injection* $\gamma_\eta\colon \Gamma_e \to \Gamma_v$ for each $\eta \in \mathrm{Ends}(e, v)$. The fundamental group of Γ is denoted $\pi_1\Gamma$, and it acts on the Bass-Serre tree T. The definitions of $\pi_1\Gamma$, of T, and of the action may be given topologically as in [SW79] or directly in terms of algebra as in [Ser80] or [Bas93], the link between the two approaches being Van Kampen's theorem. Here is a brief account of the topological definitions.

For each vertex v and edge e choose a pointed, connected CW-complex X_v, X_e and an identification of the fundamental group $\pi_1 X_v$, $\pi_1 X_e$ with the respective vertex or edge group Γ_v, Γ_e; and for each end $\eta \in \mathrm{Ends}(e, v)$ choose a pointed cellular map $\xi_\eta\colon X_e \to X_v$ inducing the injection γ_η. Construct a graph of spaces X by gluing up the disjoint union of the X_v's and the products $X_e \times \overline{e}$, where for each end $\eta \in \mathrm{Ends}(e, v)$ we glue $X_e \times p_\eta$ to X_v via the gluing $(x, p_\eta) \sim \xi_\eta(x)$ for each $x \in X_e$. For each vertex v of Γ we define $\pi_1(\Gamma, v)$ to be $\pi_1(X, v_0)$ where $v_0 \in X_v$ is the base point. There is a natural quotient map $q\colon X \to \Gamma$, which induces a decomposition of X into the point inverse images $X_t = q^{-1}(t)$, $t \in \Gamma$. Let $\tilde X$ be the universal covering space of X. The components of lifts of decomposition elements of X defines a decomposition of $\tilde X$, and the corresponding decomposition space of $\tilde X$ is the tree T. Choosing a base point $\tilde v \in \tilde X$ lying over $v \in X$ determines an identification of

$\pi_1(\Gamma, v)$ with the deck transformation group of the covering map $\tilde{X} \to X$, and the action of $\pi_1(\Gamma, v)$ respects the decomposition of $\tilde{X}$ and so descends to the required action of $\pi_1\Gamma$ on T.

In §4.4 we will review briefly the construction of the Bass-Serre tree given in [Bas93].

4.2 Edge-indexed graphs

An *edge-indexing* of a connected graph Γ is a function $I\colon \mathrm{Ends}(\Gamma) \to \mathbf{Z}_+ = \{1, 2, 3, \dots\}$. The pair (Γ, I) is called an *edge-indexed graph*. Given $v \in \mathrm{Verts}(\Gamma)$, define the *valence* of v, as usual, to be the cardinality of $\mathrm{Ends}(v)$, and define the *total index* of v to be

$$\mathrm{TI}(v) = \sum_{\eta \in \mathrm{Ends}(v)} I(\eta)$$

When the edge-indexing I is understood we will sometimes drop it from the notation and simply say that Γ is an edge-indexed graph.

For example, there is a forgetful functor which associates, to each graph of groups Γ having finite index edge-to-vertex injections, an edge-indexing I such that if $\eta \in \mathrm{Ends}(e, v)$ then $I(\eta) = [\Gamma_v : \gamma_\eta(\Gamma_e)]$. In this example, $\mathrm{TI}(v)$ equals the valence of any vertex $\tilde{v}$ of the Bass-Serre tree of Γ such that $\tilde{v}$ lies over v.

To view this example in a slightly different way, let the group G act on a tree T with quotient graph $\Gamma = T/G$; we may view the quotient map $T \to \Gamma$ as a morphism of graphs, taking vertices to vertices, edges to edges, and ends to ends, as long as we *first* subdivide any edge of T which is inverted by G. Given $\eta \in \mathrm{Ends}(e, v) \subset \mathrm{Ends}(\Gamma)$, choose $\tilde{\eta} \in \mathrm{Ends}(\tilde{e}, \tilde{v}) \subset \mathrm{Ends}(T)$ lying over η, and define $I(\eta) = [\mathrm{Stab}_G(v) : \mathrm{Stab}_G(e)]$; note that $I(\eta)$ is independent of the choice of $\tilde{\eta}$.

4.3 Covering maps

Given an edge-indexed graph Γ, to take an *elementary subdivision* of Γ means to choose a subset of the edges of Γ, add a new vertex to the interior of each chosen edge, and assign index 1 to each end incident to a new vertex; each new vertex has valence 2 and total index 2. Thus, under elementary subdivision, an edge can be subdivided into at most two edges. A general *subdivision* of Γ is the result of a finite sequence of elementary subdivisions; now an edge can be subdivided into an arbitrary finite number of edges.

Let Γ_1, Γ_2 be edge-indexed graphs. A continuous, surjective map $\mu\colon \Gamma_1 \to \Gamma_2$ is called a *covering map* if there exists a subdivision Γ_1' of Γ_1 such that the following holds:

Cellularity μ is a cellular map from Γ_1' to Γ_2, taking $\mathrm{Verts}(\Gamma_1')$ to $\mathrm{Verts}(\Gamma_2)$, and taking each edge e of Γ_1' homeomorphically to an edge $\mu(e)$ of Γ_2. There is therefore an induced map $\mu\colon \mathrm{Ends}(\Gamma_1') \to \mathrm{Ends}(\Gamma_2)$.

Subdivision normalization Given an edge e of Γ_1 and a vertex v of Γ_1' in the interior of e, if e', e'' are the edges of Γ_1' incident to v then $\mu(e') = \mu(e'')$.

Even covering Each end $\eta \in \mathrm{Ends}(\Gamma_2)$ is *evenly covered by* μ, which means: letting $w \in \mathrm{Verts}(\Gamma_2)$ be the vertex to which η is attached, for each $v \in \mathrm{Verts}(\Gamma_1')$ such

that $v \in \mu^{-1}(w)$, we have

$$I(\eta) = \sum_{\eta' \in \mathrm{Ends}(v) \cap \mu^{-1}(\eta)} I(\eta')$$

Here are a few more properties which follow immediately from the definitions:

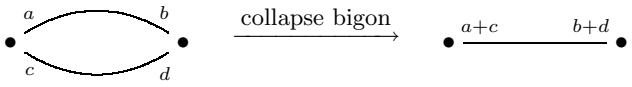
Total index preserved For each vertex v of Γ_1', we have $\mathrm{TI}(v) = \mathrm{TI}(\mu(w))$.

Folding of subdivision vertices Let v be a subdivision vertex of Γ_1', and so $\mathrm{TI}(v) = 2$ and $\mathrm{TI}(\mu(v)) = 2$. By subdivision normalization it follows that $\mu(v)$ has valence 1, with one incident end of index 2.

This last property, which derives from subdivision normalization, is needed to avoid unnecessary subdivision. As we shall see below in Lemma 11 it follows in generic cases that the subdivision Γ_1' is, in fact, just an elementary subdivision of Γ_1.

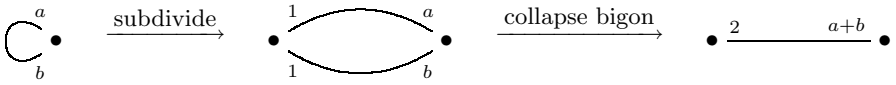
Here are some examples.

If there is a pair of vertices $v, w \in \mathrm{Verts}(\Gamma)$ and a pair of edges $e \neq e'$ each of whose two ends are attached respectively to v, w, then there is a covering map which identifies e to e' homeomorphically, and leaves the rest of Γ unchanged. We'll refer to this covering map as *collapsing a bigon*:



This covering map is defined even when $v = w$.

If e is a loop of Γ then there is a covering map which first subdivides e and then collapses the resulting bigon, leaving the rest of Γ unchanged. This covering map is called *folding a loop*:



If Γ is a finite edge-indexed graph without loops or bigons, and if any two vertices of Γ have distinct total indices, it follows that any covering map $\mu\colon \Gamma \to \Gamma'$ is an isomorphism, because μ must be one-to-one on vertices due to the fact that μ preserves total indices, and the map on vertices determines the map on edges due to the fact that Γ has no loops or bigons.

Covering maps of edge-indexed graphs arise naturally from the covering theory of graphs of groups [Bas93]. Suppose that Γ, Γ' are graphs of groups with finite index edge-to-vertex injections, and let Γ, Γ' be equipped with their natural edge indexings. Suppose that we have a covering map $\Phi\colon \Gamma \to \Gamma'$ in the graph of groups sense, as defined in [Bas93]. Then Φ is also a covering map in the edge-indexed graphs sense; this follows from Proposition 2.7 of [Bas93].

If T is an arbitrary locally finite tree and $\mathcal{G}$ is an arbitrary subgroup of $\mathrm{Isom}\, T$, then the quotient map $p\colon T \to T/\mathcal{G}$ can be regarded as a covering map with respect to a

natural edge-indexing I on $T/\mathcal{G}$, where the edge-indexing on T assigns index 1 to each end. To describe I, first subdivide any edge of T which is inverted by some element of $\mathcal{G}$, so that p is cellular. Given any vertex w of $T/\mathcal{G}$ and end $\eta \in \text{Ends}(w)$, choose $v \in p^{-1}(w)$ and $\tilde{\eta} \in p^{-1}(\eta) \cap \text{Ends}(v)$, and define $I(\eta) = [\mathcal{G}_v : \mathcal{G}_{\tilde{\eta}}]$, where $\mathcal{G}_\bullet$ denotes a stabilizer subgroup of $\mathcal{G}$; note that $I(\eta)$ is well-defined, independent of v or $\tilde{\eta}$. Note also that $I(\eta)$ is equal to the cardinality of the $\mathcal{G}_v$ orbit of $\mathcal{G}_{\tilde{\eta}}$, which is useful in proving that η is evenly covered by p.

Here is a useful construction of covering maps:

Lemma 10. *Let T be a locally finite tree, and consider subgroups $\mathcal{G} < \mathcal{G}' < \text{Isom}\, T$ and the corresponding covering maps $T \xrightarrow{p'} \Gamma' = T/\mathcal{G}'$ and $T \xrightarrow{p} \Gamma = T/\mathcal{G}$. The induced map map $\Gamma' \xrightarrow{\mu} \Gamma$ is a covering map.*

Proof. We may assume, by subdividing T if necessary, that $\mathcal{G}$ and $\mathcal{G}'$ act without edge inversions, and so the maps p, p' are cellular. Let $\mathcal{G}_\bullet, \mathcal{G}'_\bullet$ denote stabilizer subgroups of $\mathcal{G}, \mathcal{G}'$ respectively. Consider a vertex v of T, the image vertices $w = p(v)$, $w' = p'(v)$, and $\eta \in \text{Ends}(\Gamma, w)$; we must show that η is evenly covered by μ. Let $E = p^{-1}(\eta) \subset \text{Ends}(v)$, choose $\tilde{\eta} \in E$, and so the left hand side of the even covering equation for η is $[\mathcal{G}_v : \mathcal{G}_{\tilde{\eta}}] = |E|$. Let $\{\eta_1, \ldots, \eta_k\} = \mu^{-1}(\eta) \cap \text{Ends}(w)$, and let $E_i = {p'}^{-1}(\eta_i) \cap \text{Ends}(v) = {p'}^{-1}(\eta_i) \cap E$; choosing $\tilde{\eta}_i \in E_i$, the right hand side of the even covering equation for η equals the sum of $[\mathcal{G}'_v : \mathcal{G}'_{\tilde{\eta}_i}] = |E_i|$. But E is the disjoint union of $E_1, \ldots, E_k$. ♢

A finite edge-indexed graph Γ is said to be an *orbifold* if each vertex has total index 2; it follows that topologically Γ is either a circle or an arc, and each vertex is either valence 2 with two ends of index 1, or valence 1 with one end of index 2. For any covering map $\Gamma \to \Gamma'$ of edge-indexed graphs, Γ is an orbifold if and only if Γ' is an orbifold, because of the fact that total index is preserved. Covering maps between orbifolds can involve complicated subdivisions. For example, if Γ is a circle orbifold and Γ' is an arc orbifold with one edge, first do *any* subdivision of Γ resulting in an even number of edges, and then fold Γ over Γ' in zig-zag fashion. Similarly, if Γ is an arc orbifold and Γ' is an arc orbifold with one edge, first do any subdivision of Γ *whatsoever*, and then fold the result in zig-zag fashion over Γ'.

The following lemma demonstrates how subdivision normalization enforces the simplest kind of subdivision for all but the most special covering maps:

Lemma 11 (Subdivision lemma). *If $p\colon \Gamma_1 \to \Gamma_2$ is a covering map, and if Γ_2 is not an arc orbifold with one edge, then the subdivision needed to define p is an elementary subdivision; in other words, each edge of Γ_1 either maps homeomorphically to an edge of Γ_2 or is folded around an edge of Γ_2.*

Proof. Suppose that some edge e of Γ_1 is subdivided by inserting at least two distinct vertices in $\text{int}(e)$, and so Γ'_1 has an edge e' contained in the interior of e, with endpoints $a' \neq b' \in \text{int}(e)$. Consider the edge $p(e')$ of Γ_2, whose ends are located at vertices $p(a')$, $p(b')$. From the property "folding of subdivision vertices" it follows that $p(a')$ and $p(b')$ both have valence 1 and total index 2; this implies furthermore that $p(e')$ is the unique edge of Γ_2. ♢

The next lemma satisfies one's natural intuition for covering maps:

Lemma 12. *A composition of covering maps is a covering map.*

Proof. Consider a composition of covering maps $\Gamma_1 \xrightarrow{\mu_1} \Gamma_2 \xrightarrow{\mu_2} \Gamma_3$. Let Γ_1' be the subdivision needed for μ_1, and let Γ_2' be the subdivision needed for μ_2. Pulling back the subdivision points of Γ_2' defines a further subdivision Γ_1'' of Γ_1'. The map $\mu_2 \circ \mu_1$ from Γ_1'' to Γ_3 now satisfies cellularity and subdivision normalization, and even covering is easily checked. ♢

4.4 The universal covering tree

A *universal covering map* of an edge-indexed graph Γ is a covering map $\pi\colon T \to \Gamma$ such that T is a tree, regarded as an edge-indexed graph by assigning index 1 to each end of each edge of T. Every edge-indexed graph Γ has a universal covering map. This is proved in Remark 1.18 of [Bas93]; here is a construction.

Let N_v denote the regular neighborhood of v in Γ, defined as the union of the 1-cells of the barycentric subdivision of Γ that touch v. The graph N_v has the vertex v and in addition one valence 1 vertex denoted m_η corresponding to each end $\eta \in \mathrm{Ends}(v)$. Given $e \in \mathrm{Edges}(\Gamma)$ and $\eta \in \mathrm{Ends}(e)$ denote $\eta^{-1} \in \mathrm{Ends}(e)$ to be the end opposite from η, and note that $m_\eta = m_{\eta^{-1}}$. For each $v \in \mathrm{Verts}(\Gamma)$ construct a *local universal cover* $p_v\colon T_v \to N_v$, where T_v is the star on a set of cardinality $\mathrm{TI}(v)$, where p_v is a cellular map taking the star point to v, and where $\left|p_v^{-1}(m_\eta)\right| = I(\eta)$ for each $\eta \in \mathrm{Ends}(v)$. Construct T and the covering map $p\colon T \to \Gamma$ as the increasing union of subtrees $T_0 \subset T_1 \subset T_2 \subset \ldots$ and maps $p_i\colon T_i \to \Gamma$, with $p_i \mid T_j = p_j$ for $i > j$, as follows. Choose a base vertex $v \in \mathrm{Verts}(\Gamma)$, let T_0 be a disjoint copy of T_v, and let p_0 be a disjoint copy of p_v. Assuming T_i, p_i have been constructed, consider an *endpoint* $\tilde{m}$ of T_i, which means a valence 1 vertex of T_i such that $m = p_i(\tilde{m})$ is not a vertex of Γ. Let $\tilde{v}$ be the vertex of T_i closest to $\tilde{m}$, let $v = p_i(\tilde{v}) \in \mathrm{Verts}(\Gamma)$, and note that $m = m_\eta$ for some $\eta \in \mathrm{Ends}(v)$. The opposite end η^{-1} of η is located at some vertex $w \in \mathrm{Verts}(\Gamma)$. Choose a disjoint copy of T_w, and choose a point $m' \in p_w^{-1}(m)$, a valence 1 vertex of T_w. Now glue the disjoint copy of T_w to T_i by identifying $\tilde{m}$ to m'. Doing these gluings disjointly for each valence 1 vertex $\tilde{m}$ of T_i defines the tree T_{i+1}, and extending p_i by disjoint copies of the maps p_w defines the map p_{i+1}. This finishes the definition of the universal covering tree T.

Note, following Remark 1.18 of [Bas93], that the Bass-Serre tree of a graph of groups may be identified with the universal covering tree of the underlying edge-indexed graph (this holds even when edge-to-vertex injections are not of finite index, by stretching the concept of an edge-indexing to accomodate arbitrary cardinal number values for indices).

Starting from a finite valence tree T and an action of a group G on T, take the graph of groups T/G, then pass to the associated edge-indexed graph, and then take the universal covering tree; the result is naturally isomorphic to the subdivision of T obtained by elementarily subdividing each edge which is inverted by some element of G. Note in particular that if T is not a line then the full metric isometry group equals $\mathrm{Isom}\, T$, the group of cellular isometries, and so in this case if T' is the tree obtained by elementarily subdividing each edge of T that is inverted by some isometry of T then $\mathrm{Isom}\, T = \mathrm{Isom}\, T'$ and so T' is the universal covering tree of $T'/\mathrm{Isom}\, T'$.

We collect here without proof some simple facts, the first of which justifies the terminology of a "universal covering map":

Lemma 13. *An edge-indexed graph Γ with universal covering $p\colon T \to \Gamma$ satisfies the following properties:*

(1) If $f\colon \Gamma' \to \Gamma$ is a covering map and if $p'\colon T' \to \Gamma'$ is a universal covering map then there is an isomorphism between T and a subdivision of T' so that $f \circ p' = p$.

(2) If $p''\colon \Gamma \to \Gamma''$ is a covering map then the composition $T \xrightarrow{p} \Gamma \xrightarrow{p''} \Gamma''$ is a universal covering map for Γ''.

$\diamondsuit$

It follows from (1) that a *cellular* universal covering map $p\colon T \to \Gamma$ is uniquely determined by Γ up to isomorphism: if $p'\colon T' \to \Gamma$ is another cellular universal covering map then there is an isomorphism $\phi\colon T \to T'$ such that $p' \circ \phi = p$. The point here is that by definition a universal covering map $p\colon T \to \Gamma$ need not be cellular; there may be a nontrivial subdivision in the definition of p.

4.5 The geometric trichotomy

This is a simple trichotomy satisfied by the universal covering tree T of a finite, edge-indexed graph Γ:

T **is bounded;** or

T **is line-like,** meaning that there is an embedded bi-infinite line L in T and a constant $A \geq 0$ such that each point of T is a distance $\leq A$ from L; or

T **is bushy,** meaning that there is a constant $A \geq 0$ such that each point of T is a distance $\leq A$ from a vertex v with the property that $T - v$ has at least three unbounded components.

When T is line-like then T has two ends; whereas when T is bushy then its space of ends is homeomorphic to a Cantor set. The proof of this trichotomy is a standard exercise; see the comment before Lemma 14 below for an indication of a simple proof. See §5 of [BK90] for the statement and proof of the trichotomy in the case when Γ is unimodular; when Γ is not unimodular then it is easily seen that T is bushy. As noted in [BK90], in some sense this geometric trichotomy is yet another manifestation of the spherical–euclidean–hyperbolic trichotomy.

Note that the geometric trichotomy of the universal covering tree is invariant under covering maps between finite edge-indexed graphs, because if $p\colon \Gamma_1 \to \Gamma_2$ is such a covering map then the universal covering trees of Γ_1, Γ_2 are homeomorphic.

The geometric trichotomy can be detected algorithmically from a finite edge-indexed graph Γ as follows.

A *thorn* of Γ is a vertex v with total index 1; the valence must also equal 1. Equivalently, any vertex in the universal covering tree T lying over v has valence 1. To *trim* a thorn means to remove it and the incident edge, producing a smaller edge-indexed graph; the effect on the universal covering tree T is to remove the $D(\Gamma)$ orbit lying over v and the incident edges. Note that trimming does not affect the geometric trichotomy of the

universal covering tree. An edge-indexed graph Γ with no thorns is said to be *thornless*, and this happens if and only if the universal covering tree T has no valence 1 vertices, that is, if T is also thornless.

Every finite edge-indexed graph Γ can be trimmed inductively until one reaches a thornless edge-indexed graph, which can be regarded as a subgraph $\Gamma' \subset \Gamma$ called a *thornless core* of Γ.

The following simple fact is left to the reader; its proof may be used to provide a simple proof of the geometric trichotomy:

Lemma 14. *Let Γ be a finite edge-indexed graph with universal covering $p\colon T \to \Gamma$. Let Γ' be a thornless core and let $T' = p^{-1}(\Gamma')$. Then T' is a $D(\Gamma)$-invariant subtree of T, and each point of T is a uniformly bounded distance from some point of T'. Moreover:*

(1) T is bounded $\iff$ T' is a single point $\iff$ Γ' is a single point.

(2) T is line-like $\iff$ T' is a line $\iff$ Γ' is an orbifold.

(3) T is bushy $\iff$ T' has a vertex of valence ≥ 3 $\iff$ Γ' is neither a single point nor an orbifold.

In case (1) the thornless core may not be unique. In cases (2) and (3) the thornless core Γ' is unique. ♢

Because of this lemma we may extend the terminology "bushy" to apply to a finite edge-indexed graph Γ: we say that Γ is bushy if and only if its universal covering tree is bushy, which happens if and only if Γ has a unique thornless core Γ', and Γ' is neither a point nor an orbifold. Many of the unpleasant properties we have begun to encounter when the universal covering tree is a line may be simply avoided by assuming bushiness.

It should be clear that a "generic" finite edge-indexed graph is bushy, because generically the thornless core is neither a point nor an orbifold.

4.6 Deck transformation groups

Consider a finite, bushy edge-indexed graph Γ with universal covering $p\colon T \to \Gamma$. We assume that Γ has a geodesic metric which lifts to a geodesic metric on T so that each edge of T has length 1. Recall that $\operatorname{Isom} T$ denotes the topological group of cellular isometries of the tree T. The *deck transformation group* of $p\colon T \to \Gamma$ is the closed subgroup of $\operatorname{Isom} T$ defined by

$$\begin{aligned} D(\Gamma) &= \{f \in \operatorname{Isom} T \mid p \circ f = p\} \\ &= \{f \in \operatorname{Homeo}(T) \mid p \circ f = p\} \end{aligned}$$

The equation of sets on the right hand side is a consequence of the Subdivision Lemma 11, and it is the key place where we need bushiness—the equation can fail when Γ is an orbifold and the covering map $p\colon T \to \Gamma$ needs a nonelementary subdivision on T. This equation implies that the topological quotient $T/D(\Gamma)$ is naturally identified with Γ, that is, there is a natural homeomorphism $T/D(\Gamma) \overset{\phi}{\approx} \Gamma$ so that the composition $T \to T/D(\Gamma) \overset{\phi}{\approx} \Gamma$ equals

p. Moreover, the homeomorphism ϕ induces an edge-indexed isomorphism between the natural edge-indexed structure on $T/D(\Gamma)$ and Γ.

To summarize: when Γ is finite and bushy, the edge-indexed graphs $T/D(\Gamma)$ and Γ are naturally isomorphic. We will use this identification without comment in what follows.

From finiteness of Γ it follows that $D(\Gamma)$ is cocompact in $\mathrm{Isom}\, T$.

Lemma 15. *Given a covering map of finite, bushy edge indexed graphs $\mu\colon \Gamma_1 \to \Gamma_2$, with universal coverings $p\colon T \to \Gamma_1$, $\mu \circ p\colon T \to \Gamma_2$, we have $D(\Gamma_1) \subset D(\Gamma_2)$, with equality if and only if μ is an isomorphism of edge-indexed graphs.*

Given a finite, bushy edge-indexed graph Γ with universal covering $p\colon T \to \Gamma$, and given a subgroup $\mathcal{G} < \mathrm{Isom}\, T$ with $D(\Gamma) < \mathcal{G}$, the quotient map $T \to T/\mathcal{G}$ factors as $T \xrightarrow{p} \Gamma \xrightarrow{p'} T/\mathcal{G}$ for some covering map p'.

Proof. The second part is simply a special case of Lemma 10. We leave the proof of the first part to the reader, except to verify that if $\mu\colon \Gamma_1 \to \Gamma_2$ is a nonisomorphic covering map then $D(\Gamma_1) \neq D(\Gamma_2)$. To see why this is true, let Γ_1' be the subdivision of Γ_1, and note that there must two cells $c_1 \neq c_2$ of Γ_1', either both edges or both vertices, such that $\mu(c_1) = \mu(c_2)$. Choosing cells $\tilde{c}_1, \tilde{c}_2$ of T lying over c_1, c_2 respectively, $D(\Gamma_2) - D(\Gamma_1)$ contains an isometry of T taking $\tilde{c}_1$ to $\tilde{c}_2$. ♢

Corollary 16 (Existence of a minimal subcover). *Let Γ be a finite, bushy edge-indexed graph, and let $p\colon T \to \Gamma$ be the universal covering. There exists a covering map $\nu\colon \Gamma \to T/\mathrm{Isom}\, T$ which is a* minimal subcover *for Γ, meaning that for any covering map $\mu\colon \Gamma \to \Gamma'$, there exists a covering map $\mu'\colon \Gamma' \to T/\mathrm{Isom}\, T$ such that $\nu = \mu' \circ \mu$.*

It follows easily that $D(\Gamma) = \mathrm{Isom}\, T$ if and only if Γ is its own minimal subcover.

Proof. Consider the covering map $q\colon T \to T/\mathrm{Isom}\, T$. Apply Lemma 15 to obtain a covering map factorization $T \xrightarrow{p} \Gamma \xrightarrow{\nu} T/\mathrm{Isom}\, T$ of q. Consider any covering map $\mu\colon \Gamma \to \Gamma'$, and so the composition $\mu \circ p\colon T \to \Gamma'$ is a universal covering map. We have $D(p) < D(\mu \circ p) < \mathrm{Isom}\, T$, which implies in turn that $\nu\colon \Gamma \to T/\mathrm{Isom}\, T$ factors as a product of covering maps $\Gamma \xrightarrow{\mu} \Gamma' \xrightarrow{\mu'} T/\mathrm{Isom}\, T$. ♢

To summarize this discussion, a finite, bushy edge-indexed graph Γ can be considered as an encoding of a certain locally compact group, namely the deck transformation group of the universal covering of Γ. When Γ is its own minimal subcover, then Γ encodes the entire isometry group of its universal cover. Thus, via the universal covering map we obtain a bijection between the isometry types of cocompact, bushy, bounded valence trees and the isomorphism types of finite, bushy edge-indexed graphs which are their own minimal subcovers.

4.7 Unimodularity

Given an edge-indexed graph Γ there is a canonical cocycle $\xi \in C^1(\Gamma; \mathbf{Q}_+)$, where $\mathbf{Q}_+$ denotes the group of positive rational numbers under multiplication: for each oriented edge e of Γ with positive end $\eta_+(e)$ and negative end $\eta_-(e)$, define $\xi(e) = I(\eta_+(e))/I(\eta_-(e))$.

Theorem 17 ([BK90]). *Let T be a bounded valence, cocompact tree. The following are equivalent:*

- $\mathrm{Isom}\, T$ *has a discrete, cocompact subgroup.*
- $\mathrm{Isom}\, T$ *is unimodular (that is, each left invariant Haar measure is also right invariant).*
- *The cohomology class of the canonical cocycle ξ of $T/\mathrm{Isom}\, T$ is trivial.*

$\diamondsuit$

Because of this theorem, a finite edge-indexed graph Γ is called *unimodular* if the canonical cocycle ξ is cohomologically trivial. For example if Γ is a tree then it is unimodular.

5 Pumping up the deck transformation group

Throughout this section and until further notice, *all edge-indexed graphs are finite and bushy*, allowing us to apply the results of §4.6.

Let Γ be a thornless edge-indexed graph with universal cover T. We describe several methods for "pumping up" the group $D(\Gamma)$, embedding it nonsurjectively in a larger deck transformation group. More precisely, *pumping up* $D(\Gamma)$ means constructing a continuous, proper, cocompact, *nonsurjective* monomorphism $\Psi\colon D(\Gamma) \to D(\Gamma')$ where Γ' is another thornless edge-indexed graph. We describe several explicit pumping up operations. One such operation has already been discussed, namely "passing to a proper subcover", which detects when the embedding $D(\Gamma) < \mathrm{Isom}(T)$ fails to be surjective, and which allows one to construct another edge-indexed graph whose deck transformation group *is* all of $\mathrm{Isom}(T)$. Another pumping up operation, called "index 1 collapse", may apply even when $D(\Gamma) = \mathrm{Isom}(T)$, and it is particularly useful for demonstrating that T is not maximally symmetric. In addition we combine these two operations into some composite pumping up operations.

Given an edge-indexed graph Γ which is its own minimal subcover, so that $D(\Gamma) = \mathrm{Isom}(T)$, it follows that if Γ can be pumped up then T fails to be maximally symmetric. Later on we will prove the converse, giving the desired finitistic characterization of maximally symmetric trees.

5.1 Pumping up operations

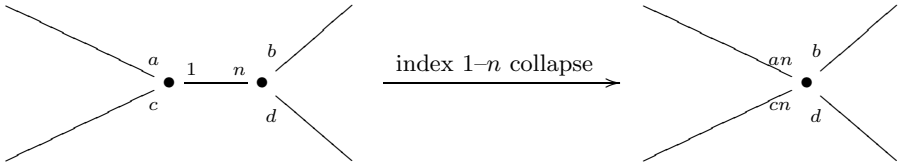
Proper subcovers Passing to a proper subcover has already been discussed:

Lemma 18 (Pumping up with proper subcovers). *For any proper covering map $\mu\colon \Gamma \to \Gamma'$, the inclusion map $D(\Gamma) \subset D(\Gamma')$ (as subgroups of $\mathrm{Isom}\, T = \mathrm{Isom}\, T'$) is continuous, proper, cocompact, and nonsurjective, and so $D(\Gamma)$ pumps up to $D(\Gamma')$.* $\diamondsuit$

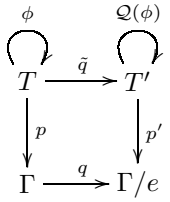
Index 1 collapse An *index 1 edge* of Γ is an edge e having an end η of index 1; let ζ be the opposite end of e. Suppose in addition that e is not a loop (index 1 collapse on e is *not defined* when e is a loop). The ends η, ζ are located at distinct vertices $v \neq w$. Letting $n = I(\zeta)$, we define the *index 1–n collapse on e*, also called an *index 1 collapse* when the value of n is unimportant, to be the edge-indexed graph Γ/e defined as follows. The underlying graph of Γ/e is obtained from Γ by collapsing the edge e to a single vertex z. The quotient map $q\colon \Gamma \to \Gamma/e$ induces a bijection $\mathrm{Ends}(\Gamma) - \{\eta, \zeta\} \leftrightarrow \mathrm{Ends}(\Gamma/e)$ denoted $\tilde{\epsilon} \leftrightarrow \epsilon$. Define the index of each $\epsilon \in \mathrm{Ends}(\Gamma/e)$ as follows:

$$I(\epsilon) = \begin{cases} I(\tilde{\epsilon}) & \text{if } \tilde{\epsilon} \notin \mathrm{Ends}(v) \\ I(\tilde{\epsilon}) \cdot I(\zeta) = I(\tilde{\epsilon}) \cdot n & \text{if } \tilde{\epsilon} \in \mathrm{Ends}(v) \end{cases}$$

For example:



Given an index 1–n collapse $q\colon \Gamma \to \Gamma/e$ and universal coverings $p\colon T \to \Gamma$ and $p'\colon T' \to \Gamma/e$, the quotient map q lifts to a quotient map $\tilde{q}\colon T \to T'$ that collapses each connected component of $p^{-1}(e)$ to a point, and $\tilde{q}$ is equivariant with respect to a homomorphism $\mathcal{Q}\colon D(\Gamma) \to D(\Gamma/e)$. That is, for all $\phi \in D(\Gamma)$ we have a commutative diagram



We call $\mathcal{Q}$ the *holonomy homomorphism* of the map q, and we note that is a continuous, proper, cocompact monomorphism from $D(\Gamma)$ to $D(\Gamma/e)$.

Notice that $\tilde{q}$ is *not* an isometry. On the other hand, $\tilde{q}$ *is* a quasi-isometry, because $\tilde{q}$ is one-to-one except on the inverse image of each vertex $\tilde{z}$ of T' lying over $z = q(e) \in \mathrm{Verts}(\Gamma/e)$, and the diameter of $\tilde{q}^{-1}(\tilde{z})$ is at most 2.

Lemma 19 (Pumping up with index 1–n collapse). *Let Γ be a thornless edge-indexed graph, and let $q\colon \Gamma \to \Gamma/e$ be an index 1–n collapse with p, p', $\tilde{q}$, and $\mathcal{Q}$ as above. If $n \geq 2$ then $\mathcal{Q}$ is not surjective and so $\mathcal{Q}$ pumps up $D(\Gamma)$ to $D(\Gamma/e)$. On the other hand, if $n = 1$ then $\mathcal{Q}$ is an isomorphism.*

Proof. Let v, w be the endpoints of e incident to the ends η, ζ of index $1, n$ respectively. Let $\tilde{w} \in T$ be a lift of w. Let $e_1, \ldots, e_n \subset T$ be the incident lifts of e, with ends $\zeta_1, \ldots, \zeta_n$ lifting ζ and incident to $\tilde{w}$, and ends $\eta_1, \ldots, \eta_n$ lifting η and incident to vertices $\tilde{v}_1, \ldots, \tilde{v}_n$

lifting v, respectively. Since Γ is thornless and $I(\eta) = 1$ it follows that there is an end $\omega \in \text{Ends}(v) - \{\eta\}$. Let $I(\omega) = m$. For each $i = 1, \ldots, n$, let $\Omega_i = \{\omega_{ij} \mid j = 1, \ldots, m\}$, be the lifts of ω located at $\tilde{v}_i$, and let $\Omega = \Omega_1 \cup \cdots \cup \Omega_n$. The subgroup $\text{Stab}\,\Omega_i$ of $D(\Gamma)$ stabilizing Ω_i clearly acts as the symmetric group S_m on the set Ω_i. The subgroup $\text{Stab}\,\Omega$ acts on Ω preserving the decomposition $\Omega = \Omega_1 \coprod \cdots \coprod \Omega_n$, and so $\text{Stab}\,\Omega$ acts on Ω as the semidirect product

$$\left(\underbrace{S_m \times \cdots \times S_m}_{n \text{ times}} \right) \rtimes S_n$$

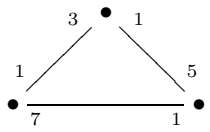
Now consider the vertex $z = q(e) \in \text{Verts}(\Gamma/e)$ and its lift $\tilde{z} = \tilde{q}(\tilde{w}) \in \text{Verts}(T')$. Note that the end $q(\omega) \in \text{Ends}(z)$ has index mn. Also, $\tilde{q}(\omega_{ij}) = \omega'_{ij} \in \text{Ends}(\tilde{z})$, and letting $\Omega'_i = \{\omega'_{ij} \mid j = 1, \ldots, m\}$ and $\Omega' = \Omega'_1 \cup \cdots \cup \Omega'_n \subset \text{Ends}(\tilde{z})$, it follows that $\text{Stab}\,\Omega' \subset D(\Gamma/e)$ acts on Ω' as the symmetric group S_{mn}.

On the other hand, clearly $\mathcal{Q}(\text{Stab}\,\Omega) \subset \text{Stab}\,\Omega'$, and the image of $\mathcal{Q}(\text{Stab}\,\Omega)$ is the subgroup of $\text{Stab}\,\Omega'$ that preserves the decomposition $\Omega' = \Omega'_1 \coprod \cdots \coprod \Omega'_n$. If $n \geq 2$ it follows that $\mathcal{Q}$ is not surjective, whereas when $n = 1$ it follows that $\mathcal{Q}$ is surjective. $\diamondsuit$

More generally, given a finite sequence of index 1 collapses $\Gamma = \Gamma_1 \to \cdots \to \Gamma_n$, the resulting composition has the effect of collapsing each component of some forest $F \subset \Gamma$ of index 1 edges; we call this an *index 1 forest collapse* on F, denoted $\Gamma \xrightarrow{q_F} \Gamma/F$, where $\Gamma/F \approx \Gamma_n$. Note that q_F pumps up the deck transformation group if and only if F contains at least one edge of index 1–n with $n \geq 2$. For every finite edge-indexed graph Γ there exists a *maximal index 1 forest collapse* $\Gamma \xrightarrow{q_F} \Gamma'$, resulting in an edge-indexed graph Γ' which has no index 1 collapse; equivalently, each index 1 edge of Γ' is a loop. Note that it is not possible to collapse an arbitrary forest of index 1 edges; for example, if a vertex v has two ends η, η' each of index 1, lying in edges e, e' respectively, and if the opposite ends of e, e' each have index ≥ 2, then at most one of e, e' is collapsed in any index 1 forest collapse. On the other hand, an arbitrary forest of index 1–1 edges can be collapsed.

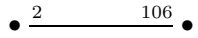
Collapse and subcover Next we define a composite operation called *collapse and subcover*. Starting from an edge indexed graph Γ, first do a maximal index 1 forest collapse, with the effect that all remaining index 1 edges are loops; then pass to the minimal subcover, which in particular has the effect of folding all loops. The resulting graph Γ' has no index 1 edges, and of course Γ' is its own minimal subcover. A collapse and subcover pumps up the deck transformation group in either of the following two situations: an index 1–n edge is collapsed for some $n \geq 2$; or the covering map is proper. A collapse and subcover fails to pump up the deck transformation group, thereby inducing an isomorphism of deck transformation groups, in the remaining case: all index 1 edges are of type 1–1, they form a forest, and collapse of this forest produces an edge-indexed graph which is its own minimal subcover.

For example, consider the edge-indexed graph



One can do an index 1 collapse on any one or two of the edges, but not on all three. An index 1 collapse on any two of these edges results in the edge-indexed graph

and folding the loop results in the graph



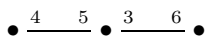
which is its own minimal subcover.

Blowup and subcover Because an index 1–1 forest collapse does not change the deck transformation group, it is possible to start with an edge-indexed graph which is its own minimal subcover, and then invert some index 1–1 forest collapse, resulting in an edge-indexed graph which may not be its own minimal subcover. We must therefore consider a composite pumping up operation called "blowup and subcover".

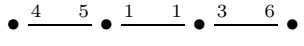
Consider a thornless edge-indexed graph Γ. A *blowup* of Γ consists of a thornless edge-indexed graph Γ' which has a forest F of type 1–1 edges, and a collapse of this forest yielding Γ, denoted $\Gamma \xleftarrow{q_F} \Gamma'$; formally q_F has the effect of identifying the collapsed graph Γ'/F isomorphically with Γ. We also require that for each vertex v of the blown up graph Γ', the valence and total index of v are not both equal to 2; this, together with thornlessness of Γ', has the important implication that Γ has only finitely many blowups up to isomorphism.

A *blowup and subcover* of Γ, denoted $\Gamma \xleftarrow{q_F} \Gamma' \xrightarrow{\mu} \Gamma''$, consists of a blowup $\Gamma \xleftarrow{q_F} \Gamma'$ followed by a minimal subcover $\Gamma' \xrightarrow{\mu} \Gamma''$; the resulting Γ'' is its own minimal subcover. A blowup and *proper* subcover pumps up the deck transformation group, because the blowup q_F induces an isomorphism $D(\Gamma) \approx D(\Gamma')$, and the proper subcover μ induces a nonsurjective monomorphism $D(\Gamma') \to D(\Gamma'')$. There are only finitely many ways to blowup and subcover Γ, up to isomorphism.

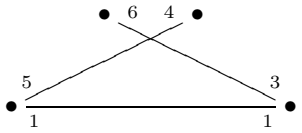
For example, consider the edge-indexed graph

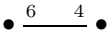


Note that this is its own minimal subcover, and so its deck transformation group equals the isometry group of its universal cover T; however, T turns out *not* to be maximally symmetric, because there is a blowup and proper subcover as follows. Blowing up the middle vertex gives



Note that the total indices are 4, 6, 4, 6 from left to right. Folding this graph up like a tri-fold wallet,



we obtain a covering map to the edge-indexed graph 

.

A similar blowup and subcover may be carried out on any edge-indexed graph of the form

5.2 Pumping up algorithm

Is it possible to pump up $D(\Gamma)$ infinitely often? At first it may seem that one can indefinitely repeat the blowup and proper subcover operation. However, we shall prove that this is impossible (a fact which depends on bushiness), leading to an algorithm which pumps up Γ as much as possible with just a few pumps.

Recall that we continue to assume all edge-indexed graphs are finite and bushy.

Lemma 20. *Let Γ be an edge-indexed graph without index 1 edges which is its own minimal subcover. There is an algorithm which constructs a blowup and subcover $\Gamma \xleftarrow{q_F} \Gamma' \xrightarrow{\mu} \Gamma''$ such that Γ'' has no index 1 edge and no blowup and proper subcover.*

The algorithm of Lemma 20 proceeds, in outline, as follows. If Γ has a blowup and proper subcover, then we show it has one $\Gamma = \Gamma_0 \leftarrow \Delta_0 \rightarrow \Gamma_1$ such that Γ_1 has no index 1 edges and is its own minimal subcover. Repeating this we obtain a sequence of blowup and proper subcovers $\Gamma_{n-1} \leftarrow \Delta_n \rightarrow \Gamma_n$ so that each Γ_n has no index 1 edges and is its own minimal subcover. This process stops at some Γ_N if and only if Γ_N has no blowup and proper subcover. To show that this eventually happens, we will prove that each Γ_n is connected to the original Γ by a single blowup and proper subcover operation $\Gamma \xleftarrow{F_n} \Gamma'_n \rightarrow \Gamma_n$, and the number of edges in the collapsing tree F_n is increasing strictly monotonically with n. The crucial fact which makes the algorithm stop is that at no stage does Γ'_n ever have a vertex of valence 2 and total index 2, which puts an upper bound on the number of edges in F_n.

Lemma 20 guarantees that the following algorithm stops:

Corollary 21 (Pumping up algorithm). *Given Γ a finite, bushy, thornless edge-indexed graph, the following algorithm pumps up $D(\Gamma)$ to $D(\Gamma')$ where Γ' is a finite, bushy edge-indexed graph without index 1 edges and with no blowup and proper subcover:*

Step 1 *Do a collapse and subcover $\Gamma \rightarrow \Gamma_1$, and so Γ_1 has no index 1 edges and is its own minimal subcover.*

Step 2 *If Γ_1 has no blowup and proper subcover, stop.*

Step 3 *Otherwise, carry out the algorithm of Lemma 20 to find a blowup and proper subcover* $\Gamma_1 \leftarrow \Gamma_2 \rightarrow \Gamma_3$ *such that* Γ_3 *has no index 1 edge and no blowup and proper subcover.*

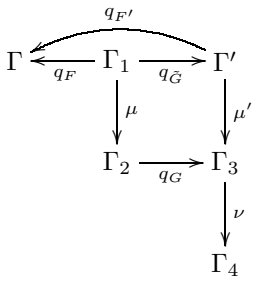
$\diamondsuit$

Proof of Lemma 20. We break the algorithm into two subroutines.

Subroutine 1 Given an edge-indexed graph Γ which is its own minimal subcover, and given any blowup and proper subcover $\Gamma \xleftarrow{q_F} \Gamma_1 \xrightarrow{\mu} \Gamma_2$, produce a blowup and proper subcover $\Gamma \xleftarrow{q_{F'}} \Gamma' \rightarrow \Gamma_4$ so that Γ_4 has no index 1 edges, and so that $F' \neq \emptyset$.

Refer to the commutative diagram below. The fact that $F' \neq \emptyset$ follows because if not then $\Gamma \approx \Gamma'$ contradicting that Γ has no proper subcovers.

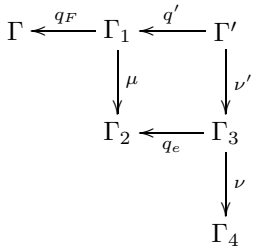
We may assume that Γ_2 does have at least one index 1 edge; note that each of them is of type 1–1, because $\mu^{-1}(e)$ for a type 1–n edge e of Γ_2 is a union of type 1–n edges of Γ_1, but each index 1 edge of Γ_1 has type 1–1. Let $\Gamma_2 \xrightarrow{q_G} \Gamma_3$ collapse a maximal forest G of index 1–1 edges. Let $\Gamma_3 \xrightarrow{\nu} \Gamma_4$ be a minimal subcover, and so Γ_4 has no index 1 edge. Consider $\tilde{G} = \mu^{-1}(G)$, a subgraph of F in Γ_1, and so $\tilde{G}$ is itself a type 1–1 forest; in fact, μ induces an isomorphism between each component of $\tilde{G}$ and a component of G. The type 1–1 forest collapse $\Gamma_1 \xrightarrow{q_F} \Gamma$ can be factored as a composition of type 1–1 forest collapses $\Gamma_1 \xrightarrow{q_{\tilde{G}}} \Gamma' \xrightarrow{q_{F'}} \Gamma$ where $F' = F/\tilde{G}$; we remark that no vertex of F has valence and total index in Γ_1 both equal to 2, and so the same is true of vertices of F' in Γ'. The covering map $\mu\colon \Gamma_1 \rightarrow \Gamma_2$ induces a covering map $\mu'\colon \Gamma' \rightarrow \Gamma_3$ so that $\mu' \circ q_{\tilde{G}} = q_G \circ \mu$. Note that properness of μ implies properness of μ': if some component of G has more than one component in its preimage under μ then this produces a point of $q_G(G)$ which has more than one component in its preimage under μ'; otherwise, some cell c of Γ which is disjoint from G has more than one preimage under μ, and it follows that $q_G(c)$ has more than one preimage under μ'. We thus obtain a blowup and proper subcover $\Gamma \xleftarrow{q_{F'}} \Gamma' \xrightarrow{\nu\circ\mu'} \Gamma_4$ so that Γ_4 has no index 1 edges.



This completes the description of Subroutine 1.

Subroutine 2 Given successive blowup and subcover operations $\Gamma \xleftarrow{q_F} \Gamma_1 \xrightarrow{\mu} \Gamma_2$ and $\Gamma_2 \xleftarrow{q_G} \Gamma_3 \xrightarrow{\nu} \Gamma_4$ so that neither Γ_2 nor Γ_4 has any index 1 edge, produce another blowup and subcover $\Gamma \xleftarrow{q_{F'}} \Gamma' \rightarrow \Gamma_4$. Moreover, $|F'| \geq |F| + |G|$.

First we consider the case where $G = \{e\}$, in which case we do a pushout as follows:



As a subset of $\Gamma_1 \times \Gamma_3$, Γ' is the set of ordered pairs (x, y) such that $\mu(x) = q_e(y)$, and the maps q', ν' are projections. We denote the vertices and edges of Γ' as "tensor products". Each vertex of Γ' is of the form $V \otimes W = (V, W)$ where $V \in \text{Verts}(\Gamma_1)$, $W \in \text{Verts}(\Gamma_3)$, and $\mu(V) = q_e(W)$. There are two types of edges in Γ'. First, letting $Z = q_e(e) \in \text{Verts}(\Gamma_2)$, for each vertex $\tilde{Z} \in \text{Verts}(\Gamma_1)$ such that $\mu(\tilde{Z}) = Z$ there is an edge $\tilde{Z} \otimes e = \tilde{Z} \times e \in \text{Edges}(\Gamma')$. Second, for each edge $D \in \text{Edges}(\Gamma_1)$ and $E \neq e \in \text{Edges}(\Gamma_3)$ such that $\mu(D) = q_e(E)$, there is an edge $D \otimes E = \{(x, y) \in D \times E \mid \mu(x) = q_e(y)\}$. Edge indexing on Γ' is defined as follows: each edge of the form $\tilde{Z} \otimes e$ has both ends of index 1; and the indexing of every other edge $D \otimes E$ is obtained by pullback under the projection $D \otimes E \to D \in \text{Edges}(\Gamma_2)$.

We first check that $\nu' \colon \Gamma' \to \Gamma_3$ is a covering map, and the only thing to verify is that each end $\eta \in \text{Ends}(\Gamma_3)$ is evenly covered. Noting that ${\nu'}^{-1}(e)$ is the disjoint union of the edges $\tilde{Z} \otimes e$, it follows that each end of e is evenly covered. If η is not an end of e, let $\eta \in \text{Ends}(W)$ for $W \in \text{Verts}(\Gamma_3)$, and consider a vertex $V \otimes W \in {\nu'}^{-1}(W)$. Note that $I(\eta) = I(q_e(\eta))$. Note also that q' induces an index preserving bijection between the set ${\nu'}^{-1}(\eta) \cap \text{Ends}(V \otimes W)$ and the set $\mu^{-1}(q_e(\eta)) \cap \text{Ends}(V)$. Since $q_e(\eta)$ is evenly covered, it follows that η is evenly covered.

Next we check that $q' \colon \Gamma' \to \Gamma_1$ is a blowup of Γ_1. Setting F' to be the set of all edges of Γ' of the form $\tilde{Z} \otimes e$, note that these edges are pairwise disjoint and so form an index 1–1 forest, and q' is the map which collapses each edge $\tilde{Z} \otimes e$ to the point $\tilde{Z}$. The only important issue to resolve is whether Γ' has a vertex of valence 2 and total index 2. Suppose there is such a vertex $V \otimes W$. Since $\Gamma \xleftarrow{q_F} \Gamma_1$ is a blowup of Γ it follows that Γ_1 has no vertex of valence 2 and total index 2, which implies that one of the two edges of Γ' incident to $V \otimes W$ is an edge $\tilde{Z} \otimes e$ of F', and so $V = \tilde{Z}$ and W is an endpoint of e. The two ends of Γ' incident to $\tilde{Z} \otimes W$ are mapped distinctly to Γ_3, and since we've already proved that ν' is a covering map it follows that W has valence 2 and total index 2 in Γ_3. But $q_e \colon \Gamma_3 \to \Gamma_2$ is a blowup of Γ_2, which implies that Γ_3 has no vertices of valence 2 and total index 2, a contradiction.

Finally, we must check that the composition $q_F \circ q' \colon \Gamma' \to \Gamma$ is a blowup of Γ. Clearly $F'' = {q'}^{-1}(F) \cup F'$ is an index 1–1 forest in Γ', and the composition $q_F \circ q'$ is just collapsing of F''. And we have already checked above that Γ' has no vertex of valence 2 and total index 2. Note also that $|F''| = \left|{q'}^{-1}(F)\right| + |F'| = |F| + |F'| \geq |F| + |G|$.

We therefore have the required blowup and subcover $\Gamma \leftarrow \Gamma' \to \Gamma_4$ when G is single edge. More generally we can proceed inductively, doing successive pushouts, to obtain the blowup and subcover for general G. This completes the description of Subroutine 2.

To prove Lemma 20, consider an edge-indexed graph Γ without index 1 edges which is its own minimal subcover. If Γ has no blowup and proper subcover, the algorithm is finished. Otherwise, choose a blowup and proper subcover $\Gamma \xleftarrow{q_{F_1}} \Gamma'_1 \xrightarrow{\mu_1} \Gamma_1$, and immediately apply Subroutine 1 to obtain one for which Γ_1 has no index 1 edges and $F_1 \neq \emptyset$. Note that Γ_1 is its own minimal subcover. Now proceed inductively as follows: assume we have a sequence of blowup and proper subcover operations $\Gamma \xleftarrow{q_{F_k}} \Gamma'_k \xrightarrow{\mu_k} \Gamma_k$ for which Γ_k has no index 1 edges and is its own minimal subcover, and such that the cardinalities $|F_k|$ are strictly increasing. If Γ_k has no blowup and proper subcover then the algorithm is finished. Otherwise, choose a blowup and proper subcover $\Gamma_k \xleftarrow{q_{G_k}} \Delta_{k+1} \xrightarrow{\nu_k} \Gamma_{k+1}$, and immediately apply Subroutine 1 to obtain one for which Γ_{k+1} has no index 1 edges and $G_k \neq \emptyset$. Now apply Subroutine 2 to obtain a blowup and proper subcover $\Gamma \xleftarrow{q_{F_{k+1}}} \Gamma'_{k+1} \xrightarrow{\mu_{k+1}} \Gamma_{k+1}$, and note that $|F_{k+1}| \geq |F_k| + |G_k| > |F_k|$. As the cardinalities of the blowup trees F_k are increasing, the blowup and subcovers $\Gamma \xleftarrow{q_{F_k}} \Gamma'_k \xrightarrow{\mu_k} \Gamma_k$ are all distinct. But Γ has only finitely many blowup and subcovers, and so the algorithm must stop. $\diamondsuit$

6 Enumerating maximally symmetric trees

In this section we prove Theorem 2 and Corollary 3. For the proof we will use without comment the fact that a bounded valence, bushy, cocompact tree is index 1 normalized if and only if its quotient edge-indexed graph has no index 1 edges. We continue to assume that all edge-indexed graphs are finite and bushy.

6.1 Graphs with no blowup and proper subcover

Besides applying Theorem 8, the heart of the proof is the following:

Proposition 22. *Let Γ be an edge-indexed graph with no index 1 edge and no blowup and proper subcover, and let $p\colon T \to \Gamma$ be the universal covering. Let Γ' be an edge-indexed graph with no index 1 edge, and let $p\colon T' \to \Gamma'$ be the universal covering. If $\Psi\colon \operatorname{Isom} T \to \operatorname{Isom} T'$ is a continuous, proper, cocompact monomorphism, then there exists an isometry $\psi\colon T \to T'$ such that $\Psi = \mathrm{ad}_\psi$.*

The techniques of proof are similar to those of Bass and Lubotzky [BL94], who study situations under which a morphism between two actions of a group H on trees T, T' is actually an isometry between T and T'. In the context of Proposition 22, if one assumes in addition that Γ has all ends of index ≥ 3, and that the edge-indexed graph $T'/\operatorname{image}(\Psi)$ has all ends of index > 1, then the conclusion follows from [BL94] Corollary 4.8(d). Proposition 22 makes no assumptions about how $\Psi(\operatorname{Isom} T)$ acts on T', other than the mild assumptions of continuity, properness, and cocompactness of Ψ. Even more significant, new techniques are needed in order to handle index 2 ends of Γ.

Proof. Let $p\colon T \to \Gamma$, $p'\colon T' \to \Gamma'$, $\Psi\colon \operatorname{Isom} T \to \operatorname{Isom} T'$ be as in the statement of the proposition. We must produce an isometry $\psi\colon T \to T'$ such that $\Psi = \mathrm{ad}_\psi$, or in other words ψ is Ψ-*equivariant* meaning that for any $f \in \operatorname{Isom} T$ we have $\psi \circ f = \Psi(f) \circ \psi$.

Keeping in mind the results of Bass and Lubotzky [BL94] mentioned above, our main difficulties are to understand index 2 ends of Γ.

Note that since Γ and Γ' are their own minimal subcovers, we have $D(\Gamma) = \mathrm{Isom}\, T$, $D(\Gamma') = \mathrm{Isom}\, T'$.

We collect some facts about the tree T. Notation: the subgroups of $\mathrm{Isom}\, T$ stabilizing a vertex v and an edge e are denoted S_v, S_e respectively.

We may assume that $\mathrm{Isom}\, T$ (and similarly $\mathrm{Isom}\, T'$) acts without edge inversions, and so for each vertex v and each incident edge e we have $S_e < S_v$; if an edge is inverted by some isometry, simply subdivide the edge at its midpoint.

For each vertex $v \in \mathrm{Verts}(T)$ and each edge e of T with endpoint v and end η incident to v, we have $[S_v : S_e] = I(p(\eta))$. The group S_v acts on the edges incident to v, and it follows that $I(p(\eta))$ equals the cardinality of the S_v-orbit of e.

An edge e of T with endpoints v, w is called an *index 2 edge* if S_e has index 2 in at least one of S_v or S_w, or equivalently if the image of e in Γ has an index 2 end.

Lemma 23. *Given two edges e, e' of T, the following are equivalent:*

(a) One of $S_e, S_{e'}$ is a subgroup of the other.

(b) $S_e = S_{e'}$.

(c) Letting $e = e_0 * e_1 * \cdots * e_k = e'$ be the unique embedded edge path in T from e to e', and letting $v_i = e_{i-1} \cap e_i$, for each $i = 1, \ldots, k$ the set $\{e_{i-1}, e_i\}$ forms a single orbit (of cardinality 2) for the action of S_{v_i} on set $\mathrm{Edges}(v_i)$ *of edges incident to v_i.*

Proof. Obviously (b) implies (a).

To show that (c) implies (b) it suffices to observe that if $e \neq e'$ are both incident to a vertex v and if $\{e, e'\}$ forms an orbit of the action of S_v then $S_e = S_{e'}$.

To prove that (a) implies (c), suppose that $S_e \subset S_{e'}$ and let $e = e_0 * \cdots * e_k = e'$ be the edge path as in (c). It follows that $S_e \subset S_{e_i}$ for $i = 1, \ldots, k$. By induction on k we easily reduce to the case $k = 1$: assuming that e, e' are incident to v and $S_e \subset S_{e'}$, we must show that $\{e, e'\}$ is an S_v orbit of the action of S_v on $\mathrm{Edges}(v)$. If this is not true, then there exists an edge $e'' \neq e, e'$ incident to v such that e', e'' are in the same S_v orbit (this uses the fact that all orbits of the S_v action on $\mathrm{Edges}(v)$ have cardinality ≥ 2). Let $T_{e'}, T_{e''}$ be the closures of the components of $T - v$ containing e', e'', respectively. Any element of S_v taking e' to e'' restricts to an isomorphism $f \colon T_{e'} \to T_{e''}$. Let $F \colon T \to T$ be the isomorphism whose restriction to $T - (T_{e'} \cup T_{e''})$ is the identity, so that $F \mid T_{e'} = f$ and $F \mid T_{e''} = f^{-1}$. Then we have $F \in S_e - S_{e'}$, contradicting that $S_e \subset S_{e'}$, and therefore showing that $\{e, e'\}$ do form an S_v orbit. ♢

The condition $S_e = S_{e'}$ is obviously an equivalence relation on edges, called *stabilizer equivalence*. Condition (c) in the lemma shows that there are three types of stabilizer equivalence classes. First is a *singleton*, a class consisting of a single edge e; this occurs when $p(e)$ has no index 2 ends. Second is a *doubleton*, a pair of edges e, e' sharing an endpoint; this occurs when $p(e) = p(e')$ has exactly one end of index 2. Third is a *line* in T, which occurs when the image of the line is an edge of Γ both of whose ends have

index 2. Condition (a) shows that if e, e' are inequivalent edges then neither of $S_e, S_{e'}$ is contained in the other.

We shall now define the map $\psi\colon T \to T'$. To define ψ on $\mathrm{Verts}(T)$, note that the map $v \to S_v$ is a bijection between $\mathrm{Verts}(T)$ and the maximal compact subgroups of $\mathrm{Isom}\, T$. Pick a representative vertex v of each orbit of $\mathrm{Isom}\, T$; the subgroup $\Psi(S_v)$ of $\mathrm{Isom}\, T'$ is compact and must therefore fix some vertex of T', and we define $\psi(v)$ be any such vertex. Extend ψ to a Ψ-equivariant map $\mathrm{Verts}(T) \to \mathrm{Verts}(T')$. Now extend to a Ψ-equivariant map $\psi\colon T \to T'$ by mapping each edge of T to a constant speed geodesic in T'.

First we show that ψ is surjective. Since $\psi(T)$ is connected, nonsurjectivity of ψ would imply that there is a vertex v' of T' so that some component C of $T' - v'$ is disjoint from $\psi(T)$. Since T' is thornless, C is unbounded. But this contradicts cocompactness of $\mathrm{image}(\Psi)$ in $\mathrm{Isom}\, T'$.

Next we show that ψ is injective on $\mathrm{Verts}(T)$. If not, consider $v_1 \neq v_2 \in \mathrm{Verts}(T)$ such that $\psi(v_1) = \psi(v_2) = w \in \mathrm{Verts}(T')$. If A is the closure of the subgroup of $\mathrm{Isom}\, T$ generated by $S_{v_1} \cup S_{v_2}$, then using the fact that S_{v_1} and S_{v_2} are maximal compact subgroups, it follows that A is a closed, noncompact subgroup of $\mathrm{Isom}\, T$. Also, $\Psi(A)$ is a closed subgroup of $\mathrm{Isom}\, T'$, by properness of Ψ. But $\Psi(A)$ is contained in S_w which is compact, and so $\Psi(A)$ is compact, contradicting properness of Ψ.

Next we show that for any vertex $v \in \mathrm{Verts}(T)$, if $v \neq x \in T$ then $\psi(v) \neq \psi(x)$. Arguing by contradiction, suppose $\psi(v) = \psi(x) = w \in \mathrm{Verts}(T')$. We may assume $x \in \mathrm{int}(e)$ for some edge e of T. Arguing as above using properness of Ψ, since S_w is compact it follows that the closure of the subgroup of $\mathrm{Isom}\, T$ generated by $S_v \cup S_e$ is compact, but S_v is a maximal compact subgroup and so $S_e \subset S_v$. Let $e_1 * \cdots * e_k = e$ be the unique embedded edge path in T which starts at v and ends with e. Since S_e stabilizes v it follows that S_e stabilizes each edge in this edge path, and by applying Lemma 23 it follows that the edges $e_1, \ldots, e_k$ are all stabilizer equivalent. Obviously v is not identified with any point in the interior of e_1, and so $k \geq 2$. Let L be the stabilizer equivalence class of $e_1, \ldots, e_k$, and so either $k = 2$ and $L = e_1 * e_2$ is a doubleton, or L is a line in T; in either case we derive a contradiction.

Case 1: $L = e_1 * e_2$**.** Let S_L be the subgroup of $\mathrm{Isom}\, T$ stabilizing L. The restriction of S_L to L is a standard $\mathbf{Z}/2$ reflection on an arc, and the map $\psi\colon L \to T'$ is $\mathbf{Z}/2$ equivariant. The image $\psi(L)$ is a subtree expressed as a union of two arcs $\psi(e_1) \cup \psi(e_2)$ sharing at least one endpoint $\psi(v_1)$. By $\mathbf{Z}/2$-equivariance it follows that for $i = 1, 2$ there is a subsegment e_i' of e_i incident to v_i such that $\psi(e_1') = \psi(e_2')$, and the sets $\psi(e_1 - e_1')$, $\psi(e_2 - e_2')$ are disjoint from each other and from $\psi(e_1') = \psi(e_2')$. Moreover, $\psi(e_1 - e_1') \neq \emptyset$ if and only if $\psi(e_2 - e_2') \neq \emptyset$. But this contradicts that ψ identifies v_0 with an interior point of e_2.

Case 2: L **is a line.** Say $L = \cdots * e_{-1} * e_0 * e_1 * \cdots * e_k * e_{k+1} * \cdots$. Let $v_i = e_{i-1} \cap e_i$, $i \in \mathbf{Z}$. The restriction to L of the stabilizer of L is a standard D_∞ action on the line L. The fixed points of the reflections in D_∞ are precisely the vertices; let r_i be the reflection fixing v_i. The even vertices $\{v_{2n}\}$ form one orbit under D_∞, and the odd vertices $\{v_{2n+1}\}$ form another orbit. The image $L' = \psi(L) \subset T'$ is the subtree of T' spanned by $\psi(\mathrm{Verts}(T))$, the group D_∞ acts on L', and the map $L \xrightarrow{\psi} L'$ is D_∞ equivariant. The action of D_∞

on L' is evidently proper and cobounded, and since L' is a tree it follows that there is a D_∞-invariant line $\ell \subset L'$ on which the D_∞ action is standard. Let $w_i \in \ell$ be the fixed point of r_i. Note that for each w_i, one of the following two properties holds, and these properties are equivariant with respect to D_∞: either $w_i = \psi(v_i)$; or $\psi(v_i) \notin \ell$ and w_i is the closest point to $\psi(v_i)$ on ℓ. In either case, it is evident from this description that the arc $\overline{\psi(v_i)\psi(v_{i+1})}$ contains none of the w's except for w_i and w_{i+1}, and therefore contains none of the $\psi(v)$'s except for $\psi(v_i)$ and $\psi(v_{i+1})$. But this contradicts that $\psi(v_0)$ lies on $\overline{\psi(v_{k-1})\psi(v_k)}$.

The argument in the last paragraph gives a little more information: it shows that for any stabilizer equivalence class which is a line L, any two nonadjacent edges of L have disjoint images under ψ.

We have shown that $\psi\colon T \to T'$ does not identify any vertex of T with any other point of T.

Next we show, for any two nonadjacent edges e, e' of T, that $f(\mathrm{int}(e)) \cap f(\mathrm{int}(e')) = \emptyset$. Suppose not: there exists $x \in \mathrm{int}(e)$, $x' \in \mathrm{int}(e')$ such that $f(x) = f(x') = y$. Letting A be the closure of the subgroup of $\mathrm{Isom}\, T$ generated by $S_e \cup S_{e'}$, it follows that $\Psi(A)$ stabilizes the point y. Using properness of Ψ it follows that A is compact. This implies that A stabilizes some vertex v of T, and so $S_e \cup S_{e'} \subset A \subset S_v$. Let $e = e_0 * \cdots * e_k$ be the shortest edge path from e to v, and similarly for $e' = e'_0 * \cdots * e'_{k'}$. It follows that $S_e \subset S_{e_i}$, $i = 0, \ldots, k$ and $S_{e'} \subset S_{e'_{i'}}$, $i' = 0, \ldots, k'$. Applying Lemma 23 it follows that the stabilizer equivalence classes L and L' of e and e' contain $e_0, \ldots, e_k$ and $e'_0, \ldots, e'_{k'}$, respectively, and so L and L' are both incident to the vertex v. Also, we already know that e, e' are not stabilizer equivalent to each other because disjoint edges of a stabilizer equivalence class have disjoint images, and so $L \neq L'$. Using the description above of a stabilizer equivalence class and its image under ψ, and using the fact that $\mathrm{int}(e)$ and $\mathrm{int}(e')$ are disjoint from $\psi(\mathrm{Verts}(T))$, we may reduce to the case that $k, k' \leq 1$; and since e, e' are not adjacent at least one of k, k' is $= 1$. Consider the case where one of k, k' equals 0, say $k = 1, k' = 0$ (the other case, where $k' = k = 1$, is similar and is left to the reader). Then there must be $x_1 \in \mathrm{int}(e_1)$ such that $\psi(x) = \psi(x_1) = \psi(x') = y$ in T'. Let $w = e_0 \cap e_1$ and choose $g \in S_w$ which interchanges $e = e_0$ with e_1, and so w interchanges x with x_1. By equivariance under g, the edge $e'' = g(e')$ contains a point x'' such that $\psi(x'') = y$, and by the argument just given it follows that the stabilizer equivalence classes L', L'' of e', e'' are distinct but are adjacent to a common vertex. But this is impossible, because e', e'' are separated from each other by the edges e_0, e_1 of the line L, and the lines L, L', L'' are distinct stabilizer equivalence classes in T.

Next we show that each edge e of T contains a point denoted m_e such that $\psi(x) \neq \psi(m_e)$ for any $x \neq m_e$; we may choose the points m_e equivariantly with respect to $\mathrm{Isom}\, T$. The point m_e is called the *midpoint* of e and the closures of the two components of $e - m_e$ are called the *halves* of e. To see why m_e exists, let v, w be the endpoints of e. Let e_v be the longest subsegment of e which is identified via ψ with a subsegment of another edge incident to v, and similarly for e_w. It follows that $e_v \cap e_w = \emptyset$ for otherwise there would be edges $e', e'' \neq e$ incident to v, w such that $\psi(e') \cap \psi(e'') \neq \emptyset$, contradiction. We can then take m_e to be any point of $\mathrm{int}(e) - (e_v \cup e_w)$.

Now we may give a global description of the map $\psi\colon T \to T'$. Given a vertex v, define $\mathrm{Star}(v)$ to be the union, over all edges e incident to v, of the half of e containing v. Note

that $\mathrm{Star}(v)$ is invariant under S_v. The restriction of ψ to $\mathrm{Star}(v)$ is an S_v-equivariant family of partial Stallings folds [Sta91]: the half-edges forming $\mathrm{Star}(v)$ are subdivided and then folded, no two half-edges being entirely folded together. These are the only identifications made by ψ. Note that for any half-edge e incident to v, if e is partially folded with any other half-edge then all half-edges in the S_v orbit of e are partially folded together to form a single path in T'; these paths, one for each partially folded orbit of v-half-edges, may then undergo further partial foldings among each other.

The description of the map $\psi\colon T \to T'$ shows that the edge-indexed graph $\hat{\Gamma} = T'/\Psi(\mathrm{Isom}\, T)$ is a blowup of Γ: for any $v \in \mathrm{Verts}(T)$, the partial folds performed on $\mathrm{Star}(v)$ are represented downstairs in Γ by a blowup of the vertex $p(v)$ of Γ, and doing this for each vertex of Γ we obtain a blowup $\Gamma \leftarrow \hat{\Gamma}$. The induced map $\hat{\Gamma} = T'/\Psi(\mathrm{Isom}\, T) \to T'/\mathrm{Isom}\, T' = \Gamma'$ is a covering map, by Lemma 10. By hypothesis, Γ has no blowup and proper subcover, and so $\hat{\Gamma}$ is isomorphic to Γ'. Also by hypothesis, Γ' has no index 1 edges, and so the blowup is trivial and the map $\psi\colon T \to T'$ is an isometry.

This completes the proof of Proposition 22. ♢

Combining Proposition 22 with Theorem 1 we immediately have:

Corollary 24. *A bounded valence, bushy, index 1 normalized tree T is maximally symmetric if and only if its quotient edge-indexed graph $\Gamma = T/\mathrm{Isom}\, T$ has no blowup and proper subcover. It follows that the correspondence taking a tree T to its quotient graph $T/\mathrm{Isom}\, T$ sets up a bijection between isometry classes of index 1 normalized maximally symmetric trees and isomorphism classes of finite edge-indexed graphs with no index 1 edge and with no blowup and proper subcover. More generally, a bounded valence, bushy, cocompact, thornless tree T is maximally symmetric if and only if the edge-indexed graph $\Gamma = T/\mathrm{Isom}\, T$ satisfies the following: each index 1 edge is of index 1–1; the collection of these edges forms a forest F; and the collapsed graph Γ/F has no blowup and proper subcover.* ♢

6.2 Proof of Theorem 2 and Corollary 3

Fix a bounded valence, bushy tree τ. Suppose that $\mathcal{G}$ is a uniform, cobounded subgroup of τ. The inclusion map $\mathcal{G} \to \mathrm{QI}(\tau)$ factors through a map $\alpha\colon \mathcal{G} \to \widehat{\mathrm{QI}}(\tau)$ which is a cobounded quasi-action of $\mathcal{G}$ on τ. Applying Theorem 8 there is a bounded valence, bushy, cocompact tree T, a cobounded action $\phi\colon \mathcal{G} \to \mathrm{Isom}\, T$, and a quasi-conjugacy $f\colon \tau \to T$ from the quasi-action α to the action $\mathcal{G}$. By trimming thorns we may assume that T is thornless, and so the natural homomorphism $\mathrm{Isom}\, T \to \mathrm{QI}(T)$ is an embedding. Consider the edge-indexed graph $\Gamma = T/\mathrm{Isom}\, T$. Applying Corollary 21 there is a collapse and subcover $\Gamma \xrightarrow{p} \Gamma_1$, and a blowup and subcover $\Gamma_1 \xleftarrow{q} \Gamma_2 \xrightarrow{\mu} \Gamma_3$, such that Γ_3 has no index 1 edge and no blowup and proper subcover. The maps p, q, μ lift to quasi-isometries of universal covering trees $T \xrightarrow{P} T_1$, $T_1 \xleftarrow{Q} T_2$, $T_2 \xrightarrow{M} T_3$. Applying Corollary 24 the tree T_3 is maximally symmetric, and applying Theorem 1 the group $\mathrm{Isom}\, T_3$ is a maximal uniform subgroup of $\mathrm{QI}(T_3)$. The graphs Γ, Γ_1, and Γ_3 are their own minimal subcovers, and so $D(\Gamma)$, $D(\Gamma_1)$, and $D(\Gamma_3)$ equal $\mathrm{Isom}\, T$, $\mathrm{Isom}\, T_1$, and $\mathrm{Isom}\, T_3$ respectively. We have $\mathrm{ad}_f(\mathcal{G}) < \mathrm{Isom}\, T$, $\mathrm{ad}_P(\mathrm{Isom}\, T) < \mathrm{Isom}\, T_1$, $\mathrm{ad}_Q^{-1}(\mathrm{Isom}\, T_1) = D(\Gamma_2)$, and $\mathrm{ad}_M(D(\Gamma_2)) < \mathrm{Isom}\, T_3$. Letting $F = M \circ \bar{Q} \circ P \circ f\colon \tau \to T_3$, where $\bar{Q}\colon T_1 \to T_2$ is a coarse inverse

of Q, it follows that $\mathrm{ad}_F = \mathrm{ad}_M \circ \mathrm{ad}_Q^{-1} \circ \mathrm{ad}_P \circ \mathrm{ad}_f \colon \mathrm{QI}(\tau) \to \mathrm{QI}(T_3)$ is an isomorphism inducing a bijection between uniform subgroups, and $\mathrm{ad}_F^{-1}(\mathrm{Isom}\, T_3)$ is a maximal uniform subgroup of $\mathrm{QI}(\tau)$ containing $\mathcal{G}$. This proves (1). Moreover, if $\mathcal{G}$ is a maximal uniform cobounded subgroup of τ it follows that $\mathcal{G} = \mathrm{ad}_F^{-1}(\mathrm{Isom}\, T_3)$, proving the first sentence of (2).

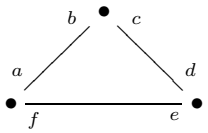
To prove the second sentence of (2), suppose that $\mathcal{G} = \mathrm{ad}_{f'}(\mathrm{Isom}\, T')$ where T' is a bounded valence, bushy, cocompact tree and $\Gamma' = T'/\mathrm{Isom}\, T'$ has no index 1 edges, and where $f' \colon T' \to \tau$ is a quasi-isometry. We thus have a quasi-isometry $T_3 \xrightarrow{H = {f'}^{-1} \circ F} T'$ with the property that ad_H takes $\mathrm{Isom}\, T_3$ into $\mathrm{Isom}\, T'$ by a continuous, proper, cocompact monomorphism. Applying Proposition 22 it follows that $\mathrm{ad}_H = \mathrm{ad}_{H'}$ for some isometry $H' \colon T_3 \to T'$, proving (2).

To prove part (3) of the theorem, suppose that $\mathcal{G}, \mathcal{G}'$ are maximal uniform subgroups of $\mathrm{QI}(\tau)$ and that $F\mathcal{G}F^{-1} = \mathcal{G}'$ for some $F \in \mathrm{QI}(\tau)$. Applying part (2) we have $\mathcal{G} = \mathrm{ad}_f(\mathrm{Isom}\, T)$, $\mathcal{G}' = \mathrm{ad}_{f'}(\mathrm{Isom}\, T')$ for some maximally symmetric trees T and quasi-isometries $f \colon T \to \tau$, $f' \colon T' \to \tau$. Also, part (2) shows that T, T' are uniquely determined up to isometry by $\mathcal{G}, \mathcal{G}'$. Moreover, part (2) shows that T, T' are isometric to each other, because the isomorphism $\mathrm{ad}_{f'}^{-1} \circ ad_F \circ \mathrm{ad}_f \colon \mathrm{Isom}(T) \to \mathrm{Isom}(T')$ is equal to ad_h for some isometry $h \colon T \to T'$. Thus, to each conjugacy class of maximal uniform subgroups of $\mathrm{QI}(\tau)$ there corresponds a well-defined isometry class of maximally symmetric trees. This correspondence is a surjection, because for every maximally symmetric tree T there exists a quasi-isometry $f \colon T \to \tau$, and so $\mathcal{G} = \mathrm{ad}_f(\mathrm{Isom}\, T)$ is a maximal uniform subgroup of $\mathrm{QI}(\tau)$ by Theorem 1. Also this correspondence is an injection, for suppose we have maximal uniform subgroups $\mathcal{G} = \mathrm{ad}_f(\mathrm{Isom}\, T), \mathcal{G}' = \mathrm{ad}_{f'}(\mathrm{Isom}\, T')$ where T, T' are maximally symmetric, $f \colon T \to \tau, f' \colon T' \to \tau$ are quasi-isometries, and T, T' are isometric; choosing an isometry $h \colon T \to T'$ it follows that $F = [f' \circ h \circ f] \in \mathrm{QI}(\tau)$ conjugates $\mathcal{G}$ to $\mathcal{G}'$.

We can prove part (4) of Theorem 2 by simply noting some examples of edge-indexed graphs with no index 1 edge nor any blowup and proper subcover.

For unimodular examples, the edge-indexed graphs $\bullet \overset{p \quad q}{\text{———}} \bullet$ with $p > q \geq 2$ clearly cannot be pumped up: they have no proper subcovers; and they have no blowups. Their universal covers give countably many isometry classes of maximally symmetric unimodular trees.

For nonunimodular examples, consider edge-indexed graphs of the form



Assign integer values ≥ 2 to a, b, c, d, e, f so that the numbers $a+f, b+c, d+e, a+1, b+1, c+1, d+1, e+1, f+1$ are pairwise unequal. These numbers are the total indices that can occur for vertices in any blowup, and it follows that each blown up graph has vertices with distinct total indices and therefore has no proper subcover. Clearly we may make the choices so that $\frac{bdf}{ace} \neq 1$ and so the edge-indexed graph is not unimodular.

See Proposition 25 below for a more satisfactory enumeration of examples.

This completes the proof of Theorem 2. ♢

Proof of Corollary 3. Let G be a virtually free group of finite rank ≥ 2. There is a finite graph of finite groups Γ' whose fundamental group is isomorphic to G [KPS73], and so G acts properly discontinuously and cocompactly on the Bass-Serre tree τ of Γ'. The kernel of the action $\phi\colon G \to \operatorname{Isom}\tau$ is finite. By Theorem 2, the group $\phi(G)$ is contained in a maximal uniform subgroup of $\mathrm{QI}(\tau)$, and that subgroup is equal to $\mathrm{ad}_f(\operatorname{Isom} T)$ for some maximally symmetric tree T and some quasi-isometry $f\colon T \to \tau$. The action $\mathrm{ad}_f^{-1} \circ \phi\colon G \to \operatorname{Isom} T$ is properly discontinuous and cocompact, and the graph of groups $\Gamma = T/\mathrm{ad}_f^{-1} \circ \phi(G)$ has fundamental group isomorphic to G and Bass-Serre tree T. ♢

Finally, we improve upon Theorem 2 part (4) as follows:

Proposition 25. *Let Γ be a finite, connected graph with no loops and no bigons and with at least one edge. Then Γ has infinitely many distinct edge-indexings with no index 1 edge and no blowup and proper subcover. Moreover, infinitely many of them are unimodular, and if Γ is not a finite tree then infinitely many of them are nonunimodular.*

Proof. One way to proceed with the proof is to construct sufficiently many examples, that is, to describe some special scheme for constructing sufficiently many edge-indexings of Γ with the desired properties. Instead we shall give a general scheme for enumerating all of the appropriate edge indexings of Γ, in effect enumerating the maximally symmetric trees T with $T/\operatorname{Isom}(T)$ isomorphic to Γ. From the description, the infinitude of appropriate edge-indexings of Γ will follow.

The enumeration scheme is carried out completely for one example Γ, after the conclusion of the proof. The reader may want to refer to this example while perusing the proof.

Instead of indexing the ends of Γ with actual numerical values, index them with variables $x_1, \ldots, x_N$. An actual edge-indexing of Γ without index 1 ends corresponds to an assignment of $(x_1, \ldots, x_N) \in \{2, 3, \ldots\}^N$.

We shall construct $X \subset \mathbf{R}^N$, a finite union of affine subspaces of $\mathbf{R}^N$ defined over $\mathbf{Q}$, such that an edge-indexing $(x_1, \ldots, x_N)$ is in X if and only if there exists a blowup and proper subcover for the edge indexing $(x_1, \ldots, x_N)$. Then, when Γ is not a finite tree, we shall construct a certain homogeneous subvariety $Y \subset \mathbf{R}^N$ of degree ≥ 3 defined over $\mathbf{Q}$, not contained in any degree 1 subvariety except for all of $\mathbf{R}^N$, such that an edge-indexing $(x_1, \ldots, x_N)$ is in Y if and only if $(x_1, \ldots, x_N)$ is a unimodular. The conclusions of the theorem will quickly follow from the form of Y.

Let $\Gamma \xleftarrow{F_1} \Gamma_1, \ldots, \Gamma \xleftarrow{F_K} \Gamma_K$ denote all possible blowups of Γ, where each end of Γ_k is indexed with one of the variables $x_1, \ldots, x_N$ or with the integer 1, as follows. As usual F_k denotes a subtree of Γ_k consisting of edges of index 1–1, the cellular map $\Gamma_k \to \Gamma$ collapses each component of F_k to a point and is otherwise one-to-one, and under this collapse there is a bijection between $\operatorname{Ends}(\Gamma_k) - \operatorname{Ends}(F_k)$ and $\operatorname{Ends}(\Gamma)$. This bijection is used to index $\operatorname{Ends}(\Gamma_k) - \operatorname{Ends}(F_k)$ with the variable edge indices $x_1, \ldots, x_N$. We assume that one of these blowups, say $\Gamma \leftarrow \Gamma_1$, is actually the identity map on Γ, and so $F_1 = \emptyset$.

Given $k = 1, \ldots, K$, let $\mu_{kj}\colon \Gamma_k \to \Gamma_{kj}$, $j = 1, \ldots, J(k)$ denote all the proper subcovers of Γ_k, where Γ_{kj} is a graph with a variable y_e indexing each end $e \in \operatorname{Ends}(\Gamma_{kj})$.

The even covering equations for μ_{kj} form a system $\mathcal{E}_{kj}$ of first degree equations in the variables $\{x_1, \ldots, x_N\} \cup \{y_e \mid e \in \mathrm{Ends}(\Gamma_{kj})\}$.

We examine the even covering system $\mathcal{E}_{kj}$ more carefully. To each end $e \in \mathrm{Ends}(\Gamma_{kj})$ located at a vertex $v \in \mathrm{Verts}(\Gamma_{kj})$, and to each vertex $w \in \mathrm{Verts}(\Gamma_k) \cap \mu_{kj}^{-1}(v)$, there corresponds an even covering equation whose right hand side is y_e and whose left hand side is a sum of those variables $x_1, \ldots, x_N$ labelling ends of $\mu_{kj}^{-1}(e) \cap \mathrm{Ends}(w)$ plus an integer equal to the cardinality of $\mathrm{Ends}(F_k) \cap \mu_{kj}^{-1}(e) \cap \mathrm{Ends}(w)$; let $\mathcal{E}_{kj}^{ew}$ denote this equation. Let $\mathcal{E}_{kj}^{e}$ be the system of equations $\mathcal{E}_{kj}^{ew}$, $w \in \mu_{kj}^{-1}(v)$; and let $\mathcal{E}_{kj}$ be the system of equations $\mathcal{E}_{kj}^{e}$, $e \in \mathrm{Ends}(\Gamma_{kj})$.

If the system $\mathcal{E}_{kj}$ is inconsistent then we may discard $\Gamma \leftarrow \Gamma_k \rightarrow \Gamma_{kj}$ as a candidate blowup and proper subcover. We may characterize inconsistency of $\mathcal{E}_{kj}$ as follows. Note that for $e \neq e'$ the subsystems $\mathcal{E}_{kj}^{e}$ and $\mathcal{E}_{kj}^{e'}$ have disjoint variable sets, and so the system $\mathcal{E}_{kj}$ is consistent if and only if each of the subsystems $\mathcal{E}_{kj}^{e}$ is consistent. The subsystem $\mathcal{E}_{kj}^{e}$ is inconsistent if and only if there exist $w, w' \in \mu_{kj}^{-1}(v)$ such that each of the two sets $\mu_{kj}^{-1}(e) \cap \mathrm{Ends}(w)$, $\mu_{kj}^{-1}(e) \cap \mathrm{Ends}(w')$ lies entirely in $\mathrm{Ends}(F_k)$ but these two sets have different cardinalities.

Assume now that the system $\mathcal{E}_{kj}$ is consistent. In each subsystem $\mathcal{E}_{kj}^{e}$, since the variable y_e occurs alone on the right hand side of each equation in $\mathcal{E}_{kj}^{e}$, we may eliminate y_e; after this elimination, if there are any equations of the form (constant)=(constant) we may eliminate them as well. This produces a new system of equations $\bar{\mathcal{E}}_{kj}^{e}$ in the variables $x_1, \ldots, x_N$. The system $\bar{\mathcal{E}}_{kj}$ is the union of the systems $\bar{\mathcal{E}}_{kj}^{e}$ for $e \in \mathrm{Ends}(\Gamma_{kj})$; we call $\bar{\mathcal{E}}_{kj}$ the system of *reduced even covering equations* for μ_{kj}. Let $X_{kj} \subset \mathbf{R}^N$ denote the solution set of $\bar{\mathcal{E}}_{kj}$, an affine subspace of $\mathbf{R}^N$ defined over $\mathbf{Q}$.

We claim that if $\bar{\mathcal{E}}_{kj}$ is vacuous, meaning that $X_{kj} = \mathbf{R}^N$, then $\mu_{kj} \colon \Gamma_k \rightarrow \Gamma_{kj}$ is a graph isomorphism and so is *not* a proper subcover.

To prove the claim, assume that $\bar{\mathcal{E}}_{kj}$ is vacuous. Since the variables in distinct subsystems $\bar{\mathcal{E}}_{kj}^{e}$ are distinct, it follows that each $\bar{\mathcal{E}}_{kj}^{e}$ is vacuous. Let v be the vertex incident to e. Vacuity of $\bar{\mathcal{E}}_{kj}^{e}$ implies one of two possibilities. In the first possibility, $\mu_{kj}^{-1}(v) \subset F_k$; in this case each equation $\mathcal{E}_{kj}^{ew}$ has the form (constant)$= y_e$, and so elimination removes all these equations. In the second possibility, $\mu_{kj}^{-1}(v)$ is a single vertex w; in this case $\mathcal{E}_{kj}^{e}$ consists of the single equation $\mathcal{E}_{kj}^{ew}$, with right hand side y_e, and elimination of y_e removes this equation. In all other cases, $\bar{\mathcal{E}}_{kj}^{e}$ is nonvacuous.

Partition the set $\mathrm{Verts}(\Gamma_k)$ into two sets: V consists of all vertices incident to some variably indexed end; and V' consist of all the rest, namely those vertices incident only to ends of index 1, the interior vertices of F_k. It follows from the previous paragraph that $\mu_{kj}(V) \cap \mu_{kj}(V') = \emptyset$ and that μ_{kj} is one-to-one on V. Since Γ_k has neither loops nor bigons it follows further that μ_{kj} is one-to-one on the union of all edges with both ends in the set V. Consider now the restriction of μ_{kj} to F_k; we have already seen that μ_{kj} is one-to-one on $F_k \cap V$ and the remaining vertices of F_k are mapped disjointly from V. But this implies that μ_{kj} is one-to-one on vertices and edges of F_k. Thus we see that μ_{kj} is an isomorphism, proving the claim.

To summarize, we have shown that the edge-indexings of Γ which *do* have a blowup and proper subcover are those which lie on a finite union $X = \bigcup_{kj} X_{kj}$ of codimension ≥ 1 affine subspaces of $\mathbf{R}^N$ defined over $\mathbf{Q}$. It follows that there are infinitely many edge

indexings with integer values ≥ 2 that do not lie in X, and so infinitely many edge-indexings with no index 1 edge and no blowup and proper subcover. If Γ is a finite tree then this finishes the proof, because every edge-indexing of Γ is unimodular.

Suppose now that Γ is not a finite tree, but instead is a graph of rank R. Let $c_1, \ldots, c_R$ be simple, closed, oriented edge paths whose corresponding 1-cycles give a basis for $H_1(\Gamma)$. Let $\ell(r) \geq 3$ be the number of edges in c_r. Applying the canonical cocycle ξ to c_r we obtain an equation of the form

$$\xi(c_r) = \frac{x_{n_1} \cdot \ldots \cdot x_{n_{\ell(r)}}}{x_{n_{\ell(r)+1}} \cdot \ldots \cdot x_{n_{2\ell(r)}}}$$

where all $2\ell(r)$ of the variables are distinct. Setting this equal to 1 and clearing the denominator we thus obtain the following homogeneous equation of degree $\ell(r)$:

$$x_{n_1} \cdot \ldots \cdot x_{n_{\ell(r)}} = x_{n_{\ell(r)+1}} \cdot \ldots \cdot x_{n_{2\ell(r)}}$$

Let Y be the simultaneous solution variety of this system of equations for $r = 1, \ldots, R$; the points on Y with integer coordinates ≥ 1 are precisely the unimodular edge-indexings of Γ. Note that rational points are dense in Y.

Clearly Y has codimension ≥ 1 and so there are infinitely many edge-indexings in the complement of $X \cup Y$, that is, infinitely many nonunimodular edge-indexings which have no index 1 edge and no blowup and proper subcover.

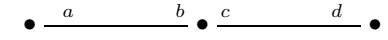
For the unimodular case, note that the homogeneous variety Y is not of degree 1, indeed Y is not contained in any linear subspace of $\mathbf{R}^N$ except for $\mathbf{R}^N$ itself. To see why we rewrite the defining equations for Y as follows. Choosing a maximal tree T of Γ we may push all variables in T to the right hand side and all variables not in T to the left hand side, and then reorder the variables, obtaining a set of defining equations for Y of the form

$$\begin{aligned} \frac{x_1}{x_2} &= f_1(x_{2R+1}, \ldots, x_N) \\ &\vdots \\ \frac{x_{2R-1}}{x_{2R}} &= f_R(x_{2R+1}, \ldots, x_N) \end{aligned}$$

where each f_r is a quotient of homogeneous monomials of equal degree $\ell(r) - 1 \geq 2$ with no variable occurring more than once in f_r. From this it is obvious that Y is not contained in any proper linear subspace of $\mathbf{R}^N$.

Decompose X as $X = X' \cup X''$ where X' is the union of those X_{kj} which are homogeneous and X'' is the union of those which are not homogeneous. As we have just seen, $Y \not\subset X'$, and it follows that the rational rays lying in $Y - X'$ are dense in Y. For each such ray ρ the intersection $\rho \cap X''$ is finite, and so assuming that ρ points into the positive orthant of $\mathbf{R}^N$ it follows that the number of integer points on $\rho - X''$ with coordinates ≥ 2 is infinite. $\diamondsuit$

An example We close with a consideration of the edge-indexed graph

Γ = 

We enumerate the thirteen different blowup and proper subcovers of Γ, writing down their reduced even covering equations. We thus obtain X as a union of thirteen affine subspaces of (a, b, c, d)-space, although it turns out that six subspaces suffice.

To help in the enumeration, consider an edge-indexed graph which is an arc with m edges. A subcover *without subdivision* of such a graph must also be an arc, because vertex valence cannot increase under a subcover. Moreover, a covering map (without subdivision) from an m-edged arc to an n-edged arc must be an m/n-folding map, under which the m edges of the domain are partitioned into m/n subarcs each with n-edges, each mapped by a graph isomorphism to the range; this follows because the ends located at a vertex of the domain must map surjectively to the ends located at the image vertex of the range. A subcover with subdivision is similarly described, with the proviso that each subdivision point must be a fold point.

We start by enumerating the proper subcovers of Γ itself, by choosing a subdivision followed by a fold.

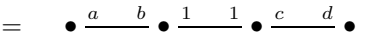
(1) Bifold Γ over a single edge; the reduced even covering equation is $a = d$.

(2) Subdivide the a–b edge and then trifold over a single edge; the equations are $a = b + c, d = 2$.

(3) Subdivide the c–d edge and trifold; the equations are $a = 2, d = b + c$.

(4) Subdivide both edges and quadrifold; the equations are $a = b + c = d$.

Next there are the proper subcovers of the unique nontrivial blowup Γ' of Γ:

Γ' = 

The proper subcovers of Γ' are:

(5) Trifold of Γ'; the equations are $a = c + 1, d = b + 1$. This subcover was depicted earlier for $(a, b, c, d) = (4, 5, 3, 6)$.

(6) Subdivide the a–b edge and quadrifold; the equations are $a = b + 1 = d, c = 1$.

(7) Subdivide the 1–1 edge and quadrifold; the equations are $a = d = 2, b = c$.

(8) Subdivide the 1–1 edge and bifold; the equations are $a = d, b = c$.

(9) Subdivide the c–d edge and quadrifold; the equations are $a = c + 1 = d, b = 1$.

(10) Subdivide the a–b and 1–1 edges and pentafold; the equations are $a = b + 1 = c + 1, d = 2$.

(11) Subdivide the a–b and c–d edges and pentafold; the equations are $a = b+1 = 2, d = c + 1 = 2$, or equivalently, $a = d = 2, b = c = 1$.

(12) Subdivide the 1–1 and c–d edges and pentafold; the equations are $a = 2, d = c+1 = b+1$.

(13) Subdivide all three edges and hexafold; the equations are $a = b+1 = c+1 = d$.

Let $X_{(i)}$ be the affine subspace defined by case (i). Then clearly $X_{(1)} = \{a = d\}$ contains $X_{(i)}$ for $i = 4, 6, 7, 8, 9, 11, 13$. Thus we may eliminate the latter seven equations, and we obtain the following set of six affine subspaces whose union is X:

$$\begin{aligned} X_{(1)} &= \{a = d\} \\ X_{(2)} &= \{a = b + c, \quad d = 2\} \\ X_{(3)} &= \{a = 2, \quad d = b + c\} \\ X_{(5)} &= \{a = c + 1, \quad d = b + 1\} \\ X_{(10)} &= \{a = b + 1 = c + 1, \quad d = 2\} \\ X_{(12)} &= \{a = 2, \quad d = c + 1 = b + 1\} \end{aligned}$$

The edge-indexings of Γ which correspond to maximally symmetric trees are precisely the quadruples (a, b, c, d) of integers ≥ 2 which lie on none of these six subspaces.

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