Eigenvalues of Non-Regular Linear Quasirandom Hypergraphs

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September 5, 2013

Abstract

In the paper "Eigenvalues and Linear Quasirandom Hypergraphs", the authors defined a spectral quasirandom property for k-uniform hypergraphs which extends the well-known graph property of the separation of the first and second largest eigenvalues of the adjacency matrix of a graph. The authors proved this spectral property is equivalent to several other hypergraph quasirandom properties, but to simplify the presentation only proved the equivalence for so-called k-uniform, coregular hypergraphs with loops. This paper extends the results of "Eigenvalues and Linear Quasirandom Hypergraphs" by proving this equivalence for all k-uniform hypergraphs, not just the coregular ones.

1 Introduction

The study of quasirandom or pseudorandom graphs was initiated by Thomason [18, 19] and then refined by Chung, Graham, and Wilson [7], resulting in a list of equivalent (deterministic) properties of graph sequences which are inspired by $G(n,p)$. Almost immediately after proving their graph theorem, Chung and Graham [2, 3, 4, 5, 6] began investigating a k-uniform hypergraph generalization. Since then, many authors have studied hypergraph quasirandomness [1, 8, 9, 12, 13, 14, 15, 16, 17].

One important k-uniform hypergraph quasirandom property is `Disc`, which states that all sufficiently large vertex sets have the same edge density as the entire hypergraph. Kohayakawa, Nagle, Rödl, and Schacht [14] and Conlon, Hàn, Person, and Schacht [8] studied `Disc` and found several properties equivalent to it, but were not able to find a generalization of a graph property called `Eig`. In graphs, `Eig` states that the first and second largest (in

*Research partly supported by NSA Grant H98230-13-1-0224.
†Research supported in part by NSF Grants 0969092 and 1300138.

absolute value) eigenvalues of the adjacency matrix are separated. The authors [16] answered this question by defining a property `Eig` for k-uniform hypergraphs and showed that it is equivalent to `Disc`, but only proved this for so-called coregular sequences. In this paper, we build up the additional algebra required to prove this equivalence for all k-uniform hypergraph sequences, not just the coregular ones. Before stating our result, we need some definitions.

Let $k \geq 2$ be an integer and let π be a proper partition of k, by which we mean that π is an unordered list of at least two positive integers whose sum is k. For the partition π of k given by $k = k_1 + \cdots + k_t$, we will abuse notation by saying that $\pi = k_1 + \cdots + k_t$. A *k-uniform hypergraph with loops* H consists of a finite set $V(H)$ and a collection $E(H)$ of k-element multisets of elements from $V(H)$. Informally, every edge has size exactly k but a vertex is allowed to be repeated inside of an edge. If F and G are k-uniform hypergraphs with loops, a *labeled copy of F in H* is an edge-preserving injection $V(F) \to V(H)$, i.e. an injection $\alpha : V(F) \to V(H)$ such that if E is an edge of F, then $\{\alpha(x) : x \in E\}$ is an edge of H. The following is our main theorem.

Theorem 1. *Let $0 < p < 1$ be a fixed constant and let $\mathcal{H} = \{H_n\}_{n\to\infty}$ be a sequence of k-uniform hypergraphs with loops such that $|V(H_n)| = n$ and $|E(H_n)| \geq p\binom{n}{k}$. Let $\pi = k_1 + \cdots + k_t$ be a proper partition of k and let $\ell \geq 2$. Assume that $\mathcal{H}$ satisfies the property*

- ***Cycle**$_{4\ell}[\pi]$: the number of labeled copies of $C_{\pi,4\ell}$ in H_n is at most $p^{|E(C_{\pi,4\ell})|}n^{|V(C_{\pi,4\ell})|} + o(n^{|V(C_{\pi,4\ell})|})$, where $C_{\pi,4\ell}$ is the hypergraph cycle of type π and length 4ℓ defined in [16, Section 2].*

Then $\mathcal{H}$ satisfies the property

- ***Eig**$[\pi]$: $\lambda_{1,\pi}(H_n) = pn^{k/2} + o(n^{k/2})$ and $\lambda_{2,\pi}(H_n) = o(n^{k/2})$, where $\lambda_{1,\pi}(H_n)$ and $\lambda_{2,\pi}(H_n)$ are the first and second largest eigenvalues of H_n with respect to π, defined in Section 2.*

When Theorem 1 is combined with [16], we obtain the following theorem.

Theorem 2. *Let $0 < p < 1$ be a fixed constant and let $\mathcal{H} = \{H_n\}_{n\to\infty}$ be a sequence of k-uniform hypergraphs with loops such that $|V(H_n)| = n$ and $|E(H_n)| \geq p\binom{n}{k} + o(n^k)$. Let $\pi = k_1 + \cdots + k_t$ be a proper partition of k. The following properties are equivalent:*

- ***Eig**$[\pi]$: $\lambda_{1,\pi}(H_n) = pn^{k/2} + o(n^{k/2})$ and $\lambda_{2,\pi}(H_n) = o(n^{k/2})$, where $\lambda_{1,\pi}(H_n)$ and $\lambda_{2,\pi}(H_n)$ are the first and second largest eigenvalues of H_n with respect to π, defined in Section 2.*

- ***Expand**$[\pi]$: For all $S_i \subseteq \binom{V(H_n)}{k_i}$ where $1 \leq i \leq t$,*

$$e(S_1, \ldots, S_t) = p\prod_{i=1}^{t} |S_i| + o\left(n^k\right)$$

  where $e(S_1, \ldots, S_t)$ is the number of tuples $(s_1, \ldots, s_t)$ such that $s_1 \cup \cdots \cup s_t$ is a hyperedge and $s_i \in S_i$.

- *$\texttt{Count}[\pi$-linear]: If F is an f-vertex, m-edge, k-uniform, π-linear hypergraph, then the number of labeled copies of F in H_n is $p^m n^f + o(n^f)$. The definition of π-linear appears in [16, Section 1].*

- *$\texttt{Cycle}_4[\pi]$: The number of labeled copies of $C_{\pi,4}$ in H_n is at most $p^{|E(C_{\pi,4})|} n^{|V(C_{\pi,4})|} + o(n^{|V(C_{\pi,4})|})$, where $C_{\pi,4}$ is the hypergraph four cycle of type π which is defined in [16, Section 2].*

- *$\texttt{Cycle}_{4\ell}[\pi]$: the number of labeled copies of $C_{\pi,4\ell}$ in H_n is at most $p^{|E(C_{\pi,4\ell})|} n^{|V(C_{\pi,4\ell})|} + o(n^{|V(C_{\pi,4\ell})|})$, where $C_{\pi,4\ell}$ is the hypergraph cycle of type π and length 4ℓ defined in [16, Section 2].*

The remainder of this paper is organized as follows. Section 2 contains the definitions we will require from [16] and also an overview of the proof of Theorem 1. Section 3 contains the algebraic properties required for the proof of Theorem 1, and finally Section 4 contains the proof of Theorem 1.

2 Definitions and Overview

First, let us recall the proof of $\texttt{Cycle}_4[1+1] \Rightarrow \texttt{Eig}[1+1]$ for regular graphs. If A is the adjacency matrix of a graph G, then A^4 counts walks of length 4 in the sense that the (i, j)-th entry of A^4 is the number of walks of length 4 between i and j. The trace of A^4 is then the number of circuits of length 4 in G. The trace of a square real symmetric matrix is the sum of its eigenvalues. If G is d-regular, then the largest eigenvalue of A^4 is d^4 so if the number of circuits of length four in G is $d^4 + o(n^4)$, then the trace of A^4 is $d^4 + o(n^4)$ which implies that all the eigenvalues of A besides d are $o(n)$.

For non-regular graphs having density p, Chung, Graham, and Wilson [7] proved that in a graph sequence satisfying $\texttt{Eig}[1+1]$, the distance between the all-ones vector and the eigenvector corresponding to the largest eigenvalue is $o(1)$ (see the bottom of page 350 in [7]). The reason for this is that if A is the adjacency matrix and v the unit length eigenvector corresponding to the largest eigenvalue, then the second largest eigenvalue of A is the spectral norm of $A - \lambda_1 vv^T$. But as the proof of the (non-regular) Expander Mixing Lemma shows, $\texttt{Expand}[1+1]$ is related to a bound on the spectral norm of $A - pn\hat{1}\hat{1}^T$, where $\hat{1}$ is the all-ones vector scaled to unit length (note $n\hat{1}\hat{1}^T = J$, the all-ones matrix). Indeed, if $S, T \subseteq V(G)$ and χ_S and χ_T are the indicator vectors for S and T respectively, then $e(S, T) - p|S||T|$ is exactly $\chi_S^T(A - pn\hat{1}\hat{1}^T)\chi_T$. Chung, Graham, and Wilson [7] proved that $\left\|v - \hat{1}\right\| = o(1)$ to conclude that $A - \lambda_1 vv^T$ and $A - pn\hat{1}\hat{1}^T$ are almost the same matrix so their spectral norms are asymptotically equal, so a bound on $\lambda_2(A)$ also bounds $\left\|A - pn\hat{1}\hat{1}^T\right\|$. Proposition 3 extends this to hypergraphs, and is our main result. Before stating this proposition, we recall several definitions from [16].

Definition. (Friedman and Wigderson [10, 11]) Let H be a k-uniform hypergraph with loops. The *adjacency map of H* is the symmetric k-linear map $\tau_H : W^k \to \mathbb{R}$ defined as follows, where W is the vector space over $\mathbb{R}$ of dimension $|V(H)|$. First, for all $v_1, \dots, v_k \in V(H)$, let

$$\tau_H(e_{v_1}, \dots, e_{v_k}) = \begin{cases} 1 & \{v_1, \dots, v_k\} \in E(H), \\ 0 & \text{otherwise,} \end{cases}$$

where e_v denotes the indicator vector of the vertex v, that is the vector which has a one in coordinate v and zero in all other coordinates. We have defined the value of τ_H when the inputs are standard basis vectors of W. Extend τ_H to all the domain linearly.

Definition. Let W be a finite dimensional vector space over $\mathbb{R}$, let $\sigma : W^k \to \mathbb{R}$ be any k-linear function, and let $\vec{\pi}$ be a proper ordered partition of k, so $\vec{\pi} = (k_1, \dots, k_t)$ for some integers $k_1, \dots, k_t$ with $t \geq 2$. Now define a t-linear function $\sigma_{\vec{\pi}} : W^{\otimes k_1} \times \cdots \times W^{\otimes k_t} \to \mathbb{R}$ by first defining $\sigma_{\vec{\pi}}$ when the inputs are basis vectors of $W^{\otimes k_i}$ and then extending linearly. For each i, $B_i = \{b_{i,1} \otimes \cdots \otimes b_{i,k_i} : b_{i,j} \text{ is a standard basis vector of W}\}$ is a basis of $W^{\otimes k_i}$, so for each i, pick $b_{i,1} \otimes \cdots \otimes b_{i,k_i} \in B_i$ and define

$$\sigma_{\vec{\pi}}\left(b_{1,1} \otimes \cdots \otimes b_{1,k_1}, \dots, b_{t,1} \otimes \cdots \otimes b_{t,k_t}\right) = \sigma(b_{1,1}, \dots, b_{1,k_1}, \dots, b_{t,1}, \dots, b_{t,k_t}).$$

Now extend $\sigma_{\vec{\pi}}$ linearly to all of the domain. $\sigma_{\vec{\pi}}$ will be t-linear since σ is k-linear.

Definition. Let $W_1, \dots, W_k$ be finite dimensional vector spaces over $\mathbb{R}$, let $\|\cdot\|$ denote the Euclidean 2-norm on W_i, and let $\phi : W_1 \times \cdots \times W_k \to \mathbb{R}$ be a k-linear map. The *spectral norm of* ϕ is

$$\|\phi\| = \sup_{\substack{x_i \in W_i \\ \|x_i\|=1}} |\phi(x_1, \dots, x_k)|.$$

Definition. Let H be a k-uniform hypergraph with loops and let $\tau = \tau_H$ be the (k-linear) adjacency map of H. Let π be any (unordered) partition of k and let $\vec{\pi}$ be any ordering of π. The *largest and second largest eigenvalues of H with respect to π*, denoted $\lambda_{1,\pi}(H)$ and $\lambda_{2,\pi}(H)$, are defined as

$$\lambda_{1,\pi}(H) := \|\tau_{\vec{\pi}}\| \quad \text{and} \quad \lambda_{2,\pi}(H) := \left\| \tau_{\vec{\pi}} - \frac{k!|E(H)|}{n^k} J_{\vec{\pi}} \right\|.$$

Definition. Let $V_1, \dots, V_t$ be finite dimensional vector spaces over $\mathbb{R}$ and let $\phi, \psi : V_1 \times \cdots \times V_t \to \mathbb{R}$ be t-linear maps. The *product* of ϕ and ψ, written $\phi * \psi$, is a $(t-1)$-linear map defined as follows. Let $u_1, \dots, u_{t-1}$ be vectors where $u_i \in V_i$. Let $\{b_1, \dots, b_{\dim(V_t)}\}$ be any orthonormal basis of V_t.

$$\phi * \psi : (V_1 \otimes V_1) \times (V_2 \otimes V_2) \times \cdots \times (V_{t-1} \otimes V_{t-1}) \to \mathbb{R}$$

$$\phi * \psi(u_1 \otimes v_1, \dots, u_{t-1} \otimes v_{t-1}) := \sum_{j=1}^{\dim(V_t)} \phi(u_1, \dots, u_{t-1}, b_j)\psi(v_1, \dots, v_{t-1}, b_j)$$

Extend the map $\phi * \psi$ linearly to all of the domain to produce a $(t-1)$-linear map.

Lemma 5 shows that the maps are well defined: the map is the same for any choice of orthonormal basis by the linearity of ϕ and ψ.

Definition. Let $V_1, \dots, V_t$ be finite dimensional vector spaces over $\mathbb{R}$ and let $\phi : V_1 \times \cdots \times V_t \to \mathbb{R}$ be a t-linear map and let s be an integer $0 \le s \le t-1$. Define

$$\phi^{2^s} : V_1^{\otimes 2^s} \times \cdots \times V_{t-s}^{\otimes 2^s} \to \mathbb{R}$$

where $\phi^{2^0} := \phi$ and $\phi^{2^s} := \phi^{2^{s-1}} * \phi^{2^{s-1}}$.

Note that we only define this for exponents which are powers of two because the product $*$ is only defined when the domains of the maps are the same. An expression like $\phi^3 = \phi*(\phi*\phi)$ does not make sense because ϕ and $\phi * \phi$ have different domains. This defines the power $\phi^{2^{t-1}}$, which is a linear map $V_1^{\otimes 2^{t-1}} \to \mathbb{R}$.

Definition. Let $V_1, \dots, V_t$ be finite dimensional vector spaces over $\mathbb{R}$ and let $\phi : V_1 \times \cdots \times V_t \to \mathbb{R}$ be a t-linear map and define $A[\phi^{2^{t-1}}]$ to be the following square matrix/bilinear map. Let $u_1, \dots, u_{2^{t-2}}, v_1, \dots, v_{2^{t-2}}$ be vectors where $u_i, v_i \in V_1$.

$$A[\phi^{2^{t-1}}] : V_1^{\otimes 2^{t-2}} \times V_1^{\otimes 2^{t-2}} \to \mathbb{R}$$
$$A[\phi^{2^{t-1}}](u_1 \otimes \cdots \otimes u_{2^{t-2}}, v_1 \otimes \dots v_{2^{t-2}}) := \phi^{2^{t-1}}(u_1 \otimes v_1 \otimes u_2 \otimes v_2 \otimes \cdots \otimes u_{2^{t-2}} \otimes v_{2^{t-2}}).$$

Extend the map linearly to the entire domain to produce a bilinear map.

Lemma 7 below proves that $A[\phi^{2^{t-1}}]$ is a square symmetric real valued matrix. The following is the main result of this note.

Proposition 3. *Let $\{\psi_r\}_{r\to\infty}$ be a sequence of symmetric k-linear maps, where $\psi_r : V_r^k \to \mathbb{R}$, V_r is a vector space over $\mathbb{R}$ of finite dimension, and $\dim(V_r) \to \infty$ as $r \to \infty$. Let $\hat{1}$ denote the all-ones vector in V_r scaled to unit length and let $J : V_r^k \to \mathbb{R}$ be the k-linear all-ones map. Let π be a proper (unordered) partition of k, and assume that for every ordering $\vec{\pi}$ of π,*

$$\lambda_1(A[\psi_{\vec{\pi}}^{2^{t-1}}]) = (1+o(1))\psi\left(\hat{1}, \dots, \hat{1}\right)^{2^{t-1}},$$
$$\lambda_2(A[\psi_{\vec{\pi}}^{2^{t-1}}]) = o\left(\lambda_1(A[\psi_{\vec{\pi}}^{2^{t-1}}])\right).$$

Then for every ordering $\vec{\pi}$ of π,

$$\|\psi_{\vec{\pi}} - qJ_{\vec{\pi}}\| = o(\psi(\hat{1}, \dots, \hat{1})),$$

where $q = \dim(V_r)^{-k/2}\psi(\hat{1}, \dots, \hat{1})$.

For graphs, $A[\tau^2]$ is the adjacency matrix squared so Proposition 3 states that the spectral norm of $A - \frac{2|E(G)|}{n^2}J$ is little-o of the square root of the largest eigenvalue of A^2, exactly what is proved by Chung, Graham, and Wilson (see the bottom of page 350 in [7]). The proof appears in the next section.

3 Algebraic properties of multilinear maps

In this section we prove several algebraic facts about multilinear maps, including Proposition 3. Throughout this section, V and V_i are finite dimensional vector spaces over $\mathbb{R}$. Also in this section we make no distinction between bilinear maps and matrices, using whichever formulation is convenient. We will use a symbol $\cdot$ to denote the input to a linear map; for example, if $\phi : V_1 \times V_2 \times V_3 \to \mathbb{R}$ is a trilinear map and $x_1 \in V_1$ and $x_2 \in V_2$, then by the expression $\phi(x_1, x_2, \cdot)$ we mean the linear map from V_3 to $\mathbb{R}$ which takes a vector $x_3 \in V_3$ to $\phi(x_1, x_2, x_3)$. Lastly, we use several basic facts about tensors, all of which follow from the fact that for finite dimensional spaces, the tensor product of V and W is the vector space over $\mathbb{R}$ of dimension $\dim(V)\dim(W)$. For example, if x and y are unit length, then $x \otimes y$ is also unit length.

Lemma 4. *Let $\phi : V \to \mathbb{R}$ be a linear map. There exists a vector v such that $\phi = \langle v, \cdot \rangle$.*

Proof. v is the vector dual to ϕ in the dual of the vector space V. Alternatively, let the ith coordinate of v be $\phi(e_i)$, since then for any x,

$$\phi(x) = \phi\left(\sum \langle x, e_i\rangle e_i\right) = \sum \langle x, e_i\rangle \phi(e_i) = \sum \langle x, e_i\rangle \langle v, e_i\rangle = \langle x, v\rangle .$$

□

Lemma 5. *Let $\phi, \psi : V_1 \times \cdots \times V_t \to \mathbb{R}$ be t-linear maps. The maps $\phi * \psi$ and $A[\phi^{2^{t-1}}]$ are well defined. Also, $\phi * \psi$ is basis independent in the sense that the definition of $\phi * \psi$ is independent of the choice of orthonormal basis $b_1, \ldots, b_t$ of V_t.*

Proof. First, extending the definitions of $\phi * \psi$ and $A[\phi^{2^{t-1}}]$ linearly to the entire domain (non-simple tensors) is well defined, since ϕ and ψ are linear. That is, write each u_i and v_i in terms of some orthonormal basis and expand each tensor in $V_i \otimes V_i$ also in terms of this basis. The linearity of ϕ and ψ then shows that the definitions of $\phi * \psi$ and $A[\phi^{2^{t-1}}]$ are well defined and linear. To see basis independence of $\phi * \psi$, by Lemma 4 the linear map $\phi(u_1, \ldots, u_{t-1}, \cdot) : V_t \to \mathbb{R}$ equals $\langle u', \cdot\rangle$ for some vector u'. Similarly, $\psi(v_1, \ldots, v_t, \cdot)$ equals $\langle v', \cdot\rangle$ for some vector v'. Then

$$(\phi * \psi)(u_1 \otimes v_1, \ldots, u_{t-1} \otimes v_{t-1}) = \sum_{i=1}^{\dim(V_t)} \langle u', b_i\rangle \langle v', b_i\rangle = \langle u', v'\rangle .$$

The last equality is valid for any orthonormal basis, since the dot product of u' and v' sums the product of the ith coordinate of u' in the basis $\{b_1, \ldots, b_{\dim(V_t)}\}$ with the ith coordinate of v' in the basis $\{b_1, \ldots, b_{\dim(V_t)}\}$. □

Definition. For $s \geq 0$ and V a finite dimensional vector space over $\mathbb{R}$, define the vector space isomorphism $\Gamma_{V,s} : V^{\otimes 2^s} \to V^{\otimes 2^s}$ as follows. If $s = 0$, define $\Gamma_{V,0}$ to be the identity map. If $s \geq 1$, let $\{b_1, \ldots, b_{\dim(V)}\}$ be any orthonormal basis of V and define for all $(i_1, \ldots, i_{2^{s-1}}, j_1, \ldots, j_{2^{s-1}}) \in [\dim(V)]^{2^s}$,

$$\Gamma_{V,s}(b_{i_1} \otimes b_{j_1} \otimes \cdots \otimes b_{i_{2^{s-1}}} \otimes b_{j_{2^{s-1}}}) = b_{j_1} \otimes b_{i_1} \otimes \cdots \otimes b_{j_{2^{s-1}}} \otimes b_{i_{2^{s-1}}} . \tag{1}$$

Extend $\Gamma_{V,s}$ linearly to all of $V^{\otimes 2^s}$.

Remarks. $\Gamma_{V,s}$ is a vector space isomorphism since it restricts to a bijection of an orthonormal basis to itself. Also, it is easy to see that $\Gamma_{V,s}$ is well defined and independent of the choice of orthonormal basis, since each b_i can be written as a linear combination of an orthonormal basis $\{b'_1, \ldots, b'_{\dim(V)}\}$ and (1) can be expanded using linearity. For notational convenience, we will usually drop the subscript V and write Γ_s for $\Gamma_{V,s}$.

Lemma 6. *Let $\phi : V_1 \times \cdots \times V_t \to \mathbb{R}$ be a t-linear map, let $0 \le s \le t-1$, and let $x_1 \in V_1^{\otimes 2^s}, \ldots, x_{t-s} \in V_{t-s}^{\otimes 2^s}$. Then*

$$\phi^{2^s}(x_1, \ldots, x_{t-s}) = \phi^{2^s}(\Gamma_s(x_1), \ldots, \Gamma_s(x_{t-s})).$$

Proof. By induction on s. The base case is $s = 0$ where Γ_0 is the identity map. Expand the definition of $\phi^{2^{s+1}}$ and use induction to obtain

$$\begin{aligned}\phi^{2^{s+1}}(x_1 \otimes y_1, \ldots, x_{t-s-1} \otimes y_{t-s-1}) &= \sum_{j=1}^{\dim(V_{t-s}^{\otimes 2^s})} \phi^{2^s}(x_1, \ldots, x_{t-s-1}, b_j)\phi^{2^s}(y_1, \ldots, y_{t-s-1}, b_j) \\ &= \sum_{j=1}^{\dim(V_{t-s}^{\otimes 2^s})} \phi^{2^s}\big(\Gamma_s(x_1), \ldots, \Gamma_s(x_{t-s-1}), \Gamma_s(b_j)\big)\phi^{2^s}\big(\Gamma_s(y_1), \ldots, \Gamma_s(y_{t-s-1}), \Gamma_s(b_j)\big).\end{aligned}$$

But since Γ_s is a vector space isomorphism, $\{\Gamma_s(b_1), \ldots, \Gamma_s(b_{\dim(V_{t-s}^{\otimes 2^s})})\}$ is an orthonormal basis of $V_{t-s}^{\otimes 2^s}$. Thus Lemma 5 shows that

$$\begin{aligned}&\sum_{j=1}^{\dim(V_{t-s}^{\otimes 2^s})} \phi^{2^s}\big(\Gamma_s(x_1), \ldots, \Gamma_s(x_{t-s-1}), \Gamma_s(b_j)\big)\phi^{2^s}\big(\Gamma_s(y_1), \ldots, \Gamma_s(y_{t-s-1}), \Gamma_s(b_j)\big) \\ &\qquad = \phi^{2^{s+1}}\big(\Gamma_s(x_1) \otimes \Gamma_s(y_1), \ldots, \Gamma_s(x_{t-s-1}) \otimes \Gamma_s(y_{t-s-1})\big)\end{aligned}$$

Finally, $\Gamma_s(x_i) \otimes \Gamma_s(y_i) = \Gamma_{s+1}(x_i \otimes y_i)$ (write x_i and y_i as linear combinations, expand $\Gamma_{s+1}(x_i \otimes y_i)$ using linearity, and apply (1)). Thus $\phi^{2^{s+1}}(x_1 \otimes y_1, \ldots, x_{t-s-1} \otimes y_{t-s-1}) = \phi^{2^{s+1}}(\Gamma_{s+1}(x_1 \otimes y_1), \ldots, \Gamma_{s+1}(x_{t-s-1} \otimes y_{t-s-1}))$, completing the proof. □

Lemma 7. *Let $V_1, \ldots, V_t$ be finite dimensional vector spaces over $\mathbb{R}$. If $\phi : V_1 \times \cdots \times V_t \to \mathbb{R}$ is a t-linear map, then $A[\phi^{2^{t-1}}]$ is a square symmetric real valued matrix.*

Proof. Let $\phi : V_1 \times \cdots \times V_t \to \mathbb{R}$ be a t-linear map. $A[\phi^{2^{t-1}}]$ is a bilinear map from $V_1^{\otimes 2^{t-2}} \times V_1^{\otimes 2^{t-2}} \to \mathbb{R}$ and so is a square matrix of dimension $\dim(V_1)^{2^{t-2}}$. Lemma 6 shows that $A[\phi^{2^{t-1}}]$ is a symmetric matrix, since

$$\begin{aligned}A[\phi^{2^{t-1}}](x_1 \otimes \cdots \otimes x_{2^{t-2}}, y_1 \otimes \cdots \otimes y_{2^{t-2}}) &= \phi^{2^{t-1}}(x_1 \otimes y_1 \otimes \cdots \otimes x_{2^{t-2}} \otimes y_{2^{t-2}}) \\ &= \phi^{2^{t-1}}(\Gamma(x_1 \otimes y_1 \otimes \cdots \otimes x_{2^{t-2}} \otimes y_{2^{t-2}})) \\ &= \phi^{2^{t-1}}(y_1 \otimes x_1 \otimes \cdots \otimes y_{2^{t-2}} \otimes x_{2^{t-2}}) \\ &= A[\phi^{2^{t-1}}](y_1 \otimes \cdots \otimes y_{2^{t-2}}, x_1 \otimes \cdots \otimes x_{2^{t-2}}).\end{aligned}$$

The above equation is valid for all $x_i, y_i \in V_1$, in particular for all basis elements of V_1 which implies that $A[\phi^{2^{t-1}}](w,z) = A[\phi^{2^{t-1}}](z,w)$ for all basis vectors w, z of $V_1^{\otimes 2^{t-2}}$. Thus $A[\phi^{2^{t-1}}]$ is a square symmetric real-valued matrix. □

Lemma 8. *Let $\phi : V_1 \times \cdots \times V_t \to \mathbb{R}$ be a t-linear map and let $x_1 \in V_1, \ldots, x_t \in V_t$ be unit length vectors. Then*

$$|\phi(x_1, \ldots, x_t)|^2 \leq \left|\phi^2(x_1 \otimes x_1, \ldots, x_{t-1} \otimes x_{t-1})\right|.$$

Proof. Consider the linear map $\phi(x_1, \ldots, x_{t-1}, \cdot)$ which is a linear map from V_t to $\mathbb{R}$. By Lemma 4, there exists a vector $w \in V_t$ such that $\phi(x_1, \ldots, x_{t-1}, \cdot) = \langle w, \cdot \rangle$. Now expand out the definition of ϕ^2:

$$\phi^2(x_1 \otimes x_1, \ldots, x_{t-1} \otimes x_{t-1}) = \sum_j |\phi(x_1, \ldots, x_{t-1}, b_j)|^2 = \sum_j |\langle w, b_j \rangle|^2 = \langle w, w \rangle$$

where the last equality is because $\{b_j\}$ is an orthonormal basis of V_t. Since $\|w\| = \sqrt{\langle w, w \rangle}$,

$$\left|\phi^2(x_1 \otimes x_1, \ldots, x_{t-1} \otimes x_{t-1})\right| = |\langle w, w \rangle| = \left|\left\langle w, \frac{w}{\|w\|} \right\rangle\right|^2.$$

But since x_t is unit length and $\langle w, \cdot \rangle$ is maximized over the unit ball at vectors parallel to w (so maximized at $w/\|w\|$), $\left|\left\langle w, \frac{w}{\|w\|} \right\rangle\right| \geq |\langle w, x_t \rangle|$. Thus

$$\left|\phi^2(x_1 \otimes x_1, \ldots, x_{t-1} \otimes x_{t-1})\right| = \left|\left\langle w, \frac{w}{\|w\|} \right\rangle\right|^2 \geq |\langle w, x_t \rangle|^2 = |\phi(x_1, \ldots, x_t)|^2.$$

The last equality used the definition of w, that $\phi(x_1, \ldots, x_{t-1}, \cdot) = \langle w, \cdot \rangle$. □

Lemma 9. *Let $\phi : V_1 \times \cdots \times V_t \to \mathbb{R}$ be a t-linear map and let $x_1 \in V_1, \ldots, x_t \in V_t$ be unit length vectors. Then for $0 \leq s \leq t-1$,*

$$|\phi(x_1, \ldots, x_t)|^{2^s} \leq \left|\phi^{2^s}(\underbrace{x_1 \otimes \cdots \otimes x_1}_{2^s}, \ldots, \underbrace{x_{t-s} \otimes \cdots \otimes x_{t-s}}_{2^s})\right|$$

which implies that

$$|\phi(x_1, \ldots, x_t)|^{2^{t-1}} \leq \left|A[\phi^{2^{t-1}}](\underbrace{x_1 \otimes \cdots \otimes x_1}_{2^{t-2}}, \underbrace{x_1 \otimes \cdots \otimes x_1}_{2^{t-2}})\right|.$$

Proof. By induction on s. The base case is $s = 0$ where both sides are equal and the induction step follows from Lemma 8. By definition of $A[\phi^{2^{t-1}}]$,

$$\left|A[\phi^{2^{t-1}}](\underbrace{x_1 \otimes \cdots \otimes x_1}_{2^{t-2}}, \underbrace{x_1 \otimes \cdots \otimes x_1}_{2^{t-2}})\right| = \left|\phi^{2^{t-1}}(\underbrace{x_1 \otimes \cdots \otimes x_1}_{2^{t-1}})\right|,$$

completing the proof. □

Lemma 10. *Let $V_1, \ldots, V_t$ be vector spaces over $\mathbb{R}$ and let $\phi : V_1 \times \cdots \times V_t \to \mathbb{R}$ be a t-linear map. Then $\|\phi\|^{2^{t-1}} \leq \lambda_1(A[\phi^{2^{t-1}}])$.*

Proof. Pick $x_1, \ldots, x_t$ unit length vectors to maximize ϕ, so $\phi(x_1, \ldots, x_t) = \|\phi\|$. Then Lemma 9 shows that

$$\|\phi\|^{2^{t-1}} = |\phi(x_1, \ldots, x_t)|^{2^{t-1}} \leq \left| A[\phi^{2^{t-1}}](\underbrace{x_1 \otimes \cdots \otimes x_1}_{2^{t-2}}, \underbrace{x_1 \otimes \cdots \otimes x_1}_{2^{t-2}}) \right|$$

Since $x_1 \otimes \cdots \otimes x_1$ is unit length, the above expression is upper bounded by the spectral norm of $A[\phi^{2^{t-1}}]$. $\square$

Lemma 11. *Let $\{M_r\}_{r\to\infty}$ be a sequence of square symmetric real-valued matrices with dimension going to infinity where $\lambda_2(M_r) = o(\lambda_1(M_r))$. Let u_r be a unit length eigenvector corresponding to the largest eigenvalue in absolute value of M_r. If $\{x_r\}$ is a sequence of unit length vectors such that $\left|x_r^T M_r x_r\right| = (1 + o(1))\lambda_1(M_r)$, then*

$$\|u_r - x_r\| = o(1).$$

Consequently, for any unit length sequence $\{y_r\}$ where each y_r is perpendicular to x_r,

$$\left|y_r^T M_r y_r\right| = o(\lambda_1(M_r)).$$

Proof. Throughout this proof, the subscript r is dropped; all terms $o(\cdot)$ should be interpreted as $r \to \infty$. This exact statement was proved by Chung, Graham, and Wilson [7], although they don't clearly state it as such. We give a proof here for completeness using slightly different language but the same proof idea: if x projected onto $u^\perp$ is too big then the second largest eigenvalue is too big. Write $x = \alpha v + \beta u$ where v is a unit length vector perpendicular to u and $\alpha, \beta \in \mathbb{C}$ and $\alpha^2 + \beta^2 = 1$ (since u is an eigenvector it might have complex entries). Let $\phi(x, y) = x^T M y$ be the bilinear map corresponding to M. Since $u^T M v = \lambda_1 u^T v = \lambda_1 \langle u, v\rangle = 0$, we have $\phi(u, v) = 0$. This implies that

$$\begin{aligned}\phi(x, x) = \phi(\alpha v + \beta u, \alpha v + \beta u) &= \alpha^2\phi(v, v) + \beta^2\phi(u, u) + 2\alpha\beta\phi(u, v)\\ &= \alpha^2\phi(v, v) + \beta^2\phi(u, u).\end{aligned}$$

The second largest eigenvalue of M is the largest eigenvalue of $M - \lambda_1(M)uu^T$ which is the spectral norm of $M - \lambda_1(M)uu^T$. Thus

$$|\phi(v, v)| = |v^T M v| = |v^T(M - \lambda_1(M)uu^T)v| \leq \lambda_2(M). \tag{2}$$

Using that $\phi(u, u) = \lambda_1(M)$ and the triangle inequality, we obtain

$$|\phi(x, x)| \leq \alpha^2\lambda_2(M) + \beta^2\lambda_1(M). \tag{3}$$

Since $\alpha^2 + \beta^2 = 1$, $|\alpha|$ and $|\beta|$ are between zero and one. Combining this with (3) and $|\phi(x, x)| = (1 + o(1))\lambda_1(M)$ and $\lambda_2(M) = o(\lambda_1(M))$, we must have $|\beta| = 1 + o(1)$ which in turn implies that $|\alpha| = o(1)$. Consequently,

$$\|u - x\|^2 = \langle u - x, u - x\rangle = \langle u, u\rangle + \langle x, x\rangle - 2\langle u, x\rangle = 2 - 2\beta = o(1).$$

Now consider some y perpendicular to x and similarly to the above, write $y = \gamma w + \delta u$ for some unit length vector w perpendicular to u and $\gamma, \delta \in \mathbb{C}$ with $\gamma^2 + \delta^2 = 1$. Then

$$\phi(y, y) = \phi(\gamma w + \delta u, \gamma w + \delta u) = \gamma^2\phi(w, w) + \delta^2\phi(u, u)$$

and as in (2), we have $|\phi(w, w)| \leq \lambda_2(M)$. Thus

$$|\phi(y, y)| \leq \gamma^2\lambda_2(M) + \delta^2\lambda_1(M).$$

We want to conclude that the above expression is $o(\lambda_1(M))$. Since $\lambda_2(M) = o(\lambda_1(M))$, we must prove that $|\delta| = o(1)$ to complete the proof.

$$\delta = \langle y, u\rangle = \left\langle y, \frac{x - \alpha v}{\beta}\right\rangle = \frac{1}{\beta}\big(\langle y, x\rangle - \alpha\langle y, v\rangle\big) = \frac{-\alpha\langle y, v\rangle}{\beta}.$$

But $|\alpha| = o(1)$, $|\beta| = 1 + o(1)$, and $\|y\| = \|v\| = 1$ so $|\delta| = o(1)$ as required. □

Lemma 12. *Let $J : V_1 \times \cdots \times V_t \to \mathbb{R}$ be the all-ones map and let $\vec{1}_i$ be the all-ones vector in V_i. Then for all $x_1, \ldots, x_t$ with $x_i \in V_i$,*

$$J(x_1, \ldots, x_t) = \left\langle \vec{1}_1, x_1\right\rangle \cdots \left\langle \vec{1}_t, x_t\right\rangle. \tag{4}$$

Proof. If $x_1, \ldots, x_t$ are standard basis vectors, then the left and right hand side of (4) are the same. By linearity, (4) is then the same for all $x_1, \ldots, x_t$. □

Proof of Proposition 3. Again throughout this proof, the subscript r is dropped; all terms $o(\cdot)$ should be interpreted as $r \to \infty$. Let $\hat{1}$ denote the all-ones vector scaled to unit length in the appropriate vector space. Pick an ordering $\vec{\pi} = (k_1, \ldots, k_t)$ of π. The definition of spectral norm is independent of the choice of the ordering for the entries of $\vec{\pi}$, so $\|\psi_{\vec{\pi}} - qJ_{\vec{\pi}}\|$ is the same for all orderings. Let $w_1, \ldots, w_t$ be unit length vectors where $(\psi_{\vec{\pi}} - qJ_{\vec{\pi}})(w_1, \ldots, w_t) = \|\psi_{\vec{\pi}} - qJ_{\vec{\pi}}\|$ and write $w_i = \alpha_i y_i + \beta_i\hat{1}$ where y_i is a unit length vector perpendicular to the all-ones vector and $\alpha_i, \beta_i \in \mathbb{R}$ with $\alpha_i^2 + \beta_i^2 = 1$. Then

$$\begin{aligned}\|\psi_{\vec{\pi}} - qJ_{\vec{\pi}}\| &= (\psi_{\vec{\pi}} - qJ_{\vec{\pi}})(w_1, \ldots, w_t) = (\psi_{\vec{\pi}} - qJ_{\vec{\pi}})(\alpha_1 y_1 + \beta_1\hat{1}, \ldots, \alpha_t y_t + \beta_t\hat{1}) \\ &= \psi_{\vec{\pi}}(\alpha_1 y_1 + \beta_1\hat{1}, \ldots, \alpha_t y_t + \beta_t\hat{1}) - q\dim(V_r)^{k/2}\prod \beta_i. \end{aligned}\tag{5}$$

The last equality used that y_i is perpendicular to $\hat{1}$, so Lemma 12 implies that if y_i appears as input to $J_{\vec{\pi}}$ then the outcome is zero no matter what the other vectors are. Thus the only non-zero term involving $J_{\vec{\pi}}$ is $J_{\vec{\pi}}(\hat{1}, \ldots, \hat{1}) = \dim(V_r)^{k/2}$. Note that $\psi(\hat{1}, \ldots, \hat{1}) = \psi_{\vec{\pi}}(\hat{1}, \ldots, \hat{1})$

since the all-ones vector scaled to unit length in $V^{\otimes k_i}$ is the tensor product of the all-ones vector scaled to unit length in V. Inserting $q = \dim(V_r)^{-k/2}\psi_{\vec{\pi}}(\hat{1},\dots,\hat{1})$ in (5), we obtain

$$\|\psi_{\vec{\pi}} - qJ_{\vec{\pi}}\| = \psi_{\vec{\pi}}(\alpha_1 y_1 + \beta_1\hat{1},\dots,\alpha_t y_t + \beta_t\hat{1}) - \left(\prod \beta_i\right)\psi_{\vec{\pi}}(\hat{1},\dots,\hat{1}). \tag{6}$$

Now consider expanding $\psi_{\vec{\pi}}$ in (6) using linearity; the term $(\prod \beta_i)\psi_{\vec{\pi}}(\hat{1},\dots,\hat{1})$ cancels, so all terms include at least one y_i. We claim that each of these terms is small; the following claim finishes the proof, since $\|\psi_{\vec{\pi}} - qJ_{\vec{\pi}}\|$ is the sum of terms each of which $o(\psi(\hat{1},\dots,\hat{1}))$.

Claim: If $z_1,\dots,z_{i-1},z_{i+1},\dots,z_t$ are unit length vectors, then

$$|\psi_{\vec{\pi}}(z_1,\dots,z_{i-1},y_i,z_{i+1},\dots,z_t)| = o(\psi(\hat{1},\dots,\hat{1})).$$

Proof. Change the ordering of $\vec{\pi}$ to an ordering $\vec{\pi}'$ that differs from $\vec{\pi}$ by swapping 1 and i. Since ψ is symmetric,

$$\psi_{\vec{\pi}}(z_1,\dots,z_{i-1},y_i,z_{i+1},\dots,z_t) = \psi_{\vec{\pi}'}(y_i,z_2,\dots,z_{i-1},z_1,z_{i+1},\dots,z_t). \tag{7}$$

Therefore proving the claim comes down to bounding $\psi_{\vec{\pi}'}(y_i,z_2,\dots,z_{i-1},z_1,z_{i+1},\dots,z_t)$, which is a combination of Lemma 9 and Lemma 11 as follows. For the remainder of this proof, denote by A the matrix $A[\psi_{\vec{\pi}'}^{2^{t-1}}]$. By assumption, we have $\lambda_2(A) = o(\lambda_1(A))$ so Lemma 11 can be applied to the matrix sequence A. Next we would like to show that we can use $\hat{1}$ for x in the statement of Lemma 11; i.e. that $A(\hat{1},\hat{1}) = (1+o(1))\lambda_1(A)$. By Lemma 9 and the assumption $\lambda_1(A) = (1+o(1))\psi(\hat{1},\dots,\hat{1})^{2^{t-1}}$, we have

$$\left|\psi_{\vec{\pi}'}(\hat{1},\dots,\hat{1})\right|^{2^{t-1}} \le \left|A(\hat{1},\hat{1})\right| \le \lambda_1(A) = (1+o(1))\psi\left(\hat{1},\dots,\hat{1}\right)^{2^{t-1}}.$$

Using the definition of $\psi_{\vec{\pi}'}$, we have $\psi_{\vec{\pi}'}(\hat{1},\dots,\hat{1}) = \psi(\hat{1},\dots,\hat{1})$, which implies asymtotic equality through the above equation. In particular, $|A(\hat{1},\hat{1})| = (1+o(1))\lambda_1(A)$ which is the condition in Lemma 11 for $x = \hat{1}$. Lastly, to apply Lemma 11 we need a vector y perpendicular to $\hat{1}$. The vector $y_i \otimes \cdots \otimes y_i \in V^{\otimes k_i 2^{t-2}}$ is pependicular to $\hat{1}$ (in $V^{\otimes k_i 2^{t-2}}$) since y_i itself is perpendicular to $\hat{1}$ (in $V^{\otimes k_i}$). Thus Lemma 11 implies that

$$\left| A(\underbrace{y_i \otimes \cdots \otimes y_i}_{2^{t-2}}, \underbrace{y_i \otimes \cdots \otimes y_i}_{2^{t-2}}) \right| = o(\lambda_1(A)). \tag{8}$$

Using Lemma 9 again shows that

$$|\psi_{\vec{\pi}'}(y_i,z_2,\dots,z_{i-1},z_1,z_{i+1},\dots,z_t)|^{2^{t-1}} \le \left| A(\underbrace{y_i \otimes \cdots \otimes y_i}_{2^{t-2}}, \underbrace{y_i \otimes \cdots \otimes y_i}_{2^{t-2}}) \right|.$$

Combining this equation with (7) and (8) shows that $|\psi_{\vec{\pi}}(z_1,\dots,z_{i-1},y_i,z_{i+1},\dots,z_t)|^{2^{t-1}} = o(\lambda_1(A))$. By assumption, $\lambda_1(A) = (1+o(1))\psi(\hat{1},\dots,\hat{1})^{2^{t-1}}$, completing the proof of the claim.

□

4 $\texttt{Cycle}_{4\ell}[\pi] \Rightarrow \texttt{Eig}[\pi]$

In this section, we prove that $\texttt{Cycle}_{4\ell}[\pi] \Rightarrow \texttt{Eig}[\pi]$ for any hypergraph sequence $\mathcal{H}$ with $|V(H_n)| = n$ and $|E(H_n)| \geq p\binom{n}{k} + o(n^k)$. We require the following result from [16].

Proposition 13. *[16, Proposition 6]Let H be a k-uniform hypergraph, let $\vec{\pi}$ be a proper ordered partition of k, and let $\ell \geq 2$ be an integer. Let τ be the adjacency map of H. Then $Tr\left[A[\tau_{\vec{\pi}}^{2^{t-1}}]^{\ell}\right]$ is the number of labeled circuits of type $\vec{\pi}$ and length 2ℓ in H.*

Propositions 3 and 13 and Lemma 10 combine to prove that $\texttt{Cycle}_{4\ell}[\pi] \Rightarrow \texttt{Eig}[\pi]$ for any proper partition π as follows.

Proof that $\texttt{Cycle}_{4\ell}[\pi] \Rightarrow \texttt{Eig}[\pi]$. Let $\mathcal{H} = \{H_n\}_{n\to\infty}$ be a sequence of hypergraphs and let τ_n be the adjacency map of H_n. For notational convenience, the subscript on n is dropped below. Throughout this proof, we use $\hat{1}$ to denote the all-ones vector scaled to unit length. Wherever we use the notation $\hat{1}$, it is the input to a multilinear map and so $\hat{1}$ denotes the all-ones vector in the appropriate vector space corresponding to whatever space the map is expecting as input. This means that in the equations below $\hat{1}$ can stand for different vectors in the same expression, but attempting to subscript $\hat{1}$ with the vector space (for example $\hat{1}_{V_3}$) would be notationally awkward.

The proof that $\texttt{Cycle}_{4\ell}[\pi] \Rightarrow \texttt{Eig}[\pi]$ comes down to checking the conditions of Proposition 3. Let $\vec{\pi}$ be any ordering of the entries of π. We will show that the first and second largest eigenvalues of $A = A[\tau_{\vec{\pi}}^{2^{t-1}}]$ are separated. Let $m = |E(C_{\pi,4\ell})| = 2\ell 2^{t-1}$ and note that $|V(C_{\pi,4\ell})| = mk/2$ since $C_{\pi,4\ell}$ is two-regular. A is a square symmetric real valued matrix, so let $\mu_1, \dots, \mu_d$ be the eigenvalues of A arranged so that $|\mu_1| \geq \cdots \geq |\mu_d|$, where $d = \dim(A)$. The eigenvalues of $A^{2\ell}$ are $\mu_1^{2\ell}, \dots, \mu_d^{2\ell}$ and the trace of $A^{2\ell}$ is $\sum_i \mu_i^{2\ell}$. Since all $\mu_i^{2\ell} \geq 0$, Proposition 13 and $\texttt{Cycle}_{4\ell}[\pi]$ implies that

$$\mu_1^{2\ell} + \mu_2^{2\ell} \leq \text{Tr}\left[A^{2\ell}\right] = \#\{\text{possibly degenerate } C_{\pi,4\ell} \text{ in } H_n\} \leq p^m n^{mk/2} + o(n^{mk/2}). \tag{9}$$

We now verify the conditions on μ_1 and μ_2 in Proposition 3, and to do that we need to compute $\tau(\hat{1}, \dots, \hat{1})$. Simple computations show that

$$\tau(\hat{1}, \dots, \hat{1}) = \tau_{\vec{\pi}}(\hat{1}, \dots, \hat{1}) = \frac{k!E(H)}{n^{k/2}}. \tag{10}$$

Using that $|E(H_n)| \geq p\binom{n}{k} + o(n^k)$, Lemma 10, and $\mu_1^{2\ell} \leq p^m n^{mk/2} + o(n^{mk/2})$ from (9),

$$pn^{k/2} + o(n^{k/2}) \leq \frac{k!E(H)}{n^{k/2}} = \tau_{\vec{\pi}}(\hat{1}, \dots, \hat{1}) \leq \|\tau_{\vec{\pi}}\| \leq \mu_1^{1/2^{t-1}} \leq pn^{k/2} + o(n^{k/2}). \tag{11}$$

This implies equality up to $o(n^{k/2})$ throughout the above expression, so $\tau(\hat{1}, \dots, \hat{1}) = pn^{k/2} + o(n^{k/2})$, $\lambda_{1,\pi}(H_n) = \|\tau_{\vec{\pi}}\| = pn^{k/2} + o(n^{k/2})$, and $\mu_1 = p^{2^{t-1}} n^{k2^{t-2}} + o(n^{k2^{t-2}})$, so $\mu_1 = (1 + o(1))\tau(\hat{1}, \dots, \hat{1})^{2^{t-1}}$.

Insert $\mu_1 = p^{2^{t-1}} n^{k2^{t-2}} + o(n^{k2^{t-2}})$ into (9) to show that $\mu_2 = o(n^{k2^{t-2}})$. Therefore, the conditions of Proposition 3 are satisfied, so

$$\|\tau_{\vec{\pi}} - qJ_{\vec{\pi}}\| = o(\tau(\hat{1}, \ldots, \hat{1})) = o(n^{k/2}),$$

where $q = n^{-k/2}\tau(\hat{1}, \ldots, \hat{1})$. Using (10), $q = k!|E(H)|/n^k$. Thus $\|\tau_{\vec{\pi}} - qJ_{\vec{\pi}}\| = \lambda_{2,\pi}(H_n)$ and the proof is complete. □

The above proof can be extended to even length cycles in the case when $\vec{\pi} = (k_1, k_2)$ is a partition into two parts. For these $\vec{\pi}$, the matrix $A[\tau_{\vec{\pi}}^2]$ can be shown to be positive semidefinite since $A[\tau_{\vec{\pi}}^2]$ will equal MM^T where M is the matrix associated to the bilinear map $\tau_{\vec{\pi}}$. Since $A[\tau_{\vec{\pi}}^2]$ is positive semidefinite, each $\mu_i \geq 0$ so any power of μ_i is non-negative. For partitions into more than two parts, we don't know if the matrix $A[\tau_{\vec{\pi}}^{2^{t-1}}]$ is always positive semidefinite or not.

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