Solutions to CS 401 / MCS 401 Quiz Number #1 — Fall 2007

1. a) By Stirling's formula, $n! \approx (n/e)^n \text{sqrt}(2\pi n)$. So

$$\lim_{n\to\infty} f(n)/g(n) = \lim_{n\to\infty} 5\,(\lg(n!))^2 / 2\lg((n!)^{7n})$$

$$= \lim_{n\to\infty} 5\,(\lg(n!))^2 / 14n\lg(n!) \qquad (\lg(x^y) = y\lg(x))$$

$$= \lim_{n\to\infty} 5\,(n\lg(n))^2 / 14n^2\lg(n) \qquad \text{(Stirling's Formula)}$$

$$= \lim_{n\to\infty} 5\lg(n)\ /\ 14 = \infty. \qquad \text{(Cancel } n^2\lg(n).)$$

Therefore $\boldsymbol{f(n) = \omega(g(n))}$.

b) $$\lim_{n\to\infty} f(n)/g(n) = \lim_{n\to\infty} n^2(\lg(n))^4 / 5^{\lg(n)}$$

$$= \lim_{n\to\infty} n^2(\lg(n))^4 / n^{\lg(5)} \qquad (x^{\lg(y)} = y^{\lg(x)}.)$$

$$= \lim_{n\to\infty} (\lg(n))^4 / n^{\lg(5)-2}$$

$$= 0 \qquad (\lg(5)-2>0,\ \text{polynomials dominate logarithms.})$$

Therefore $\boldsymbol{f(n) = o(g(n))}$.

c) $$\lim_{n\to\infty} f(n)/g(n) = \lim_{n\to\infty} (4^n n^2 + 3^n n^6) / 4^{n-2} n^2$$

$$= \lim_{n\to\infty} \left(16 + 16(3/4)^n n^4\right)$$

$$= 16 + 16\cdot\lim_{n\to\infty} (3/4)^n n^4$$

$$= 16 \qquad \text{(Exponentials dominate polynomials.)}$$

Therefore $\boldsymbol{f(n) = \Theta(g(n))}$.

2. a) The running time of algorithm A on the current computer is roughly cn^3 for some constant c. We are given $c\cdot 1000^3 = 1\,min$, so $c = 1/1000^3\ min$. The old computer is 1/8 as fast as the current one, so the running time on the old computer wil be $8cn^3\ min = (8/1000^3)n^3\ min$. The time on the old machine will be 1 *min* if $(8/1000^3)n^3 = 1$, or $n^3 = 1000^3 / 8$, or $\boldsymbol{n = 1000 / 2 = 500}$.

b) The running time of algorithm B on the current computer is roughly $c2^n$ for some constant c. We are given $c\cdot 2^{40} = 1\,min$, so $c = 2^{-40}\ min$. The old computer is 1/8 as fast as the current one, so the running time on the old computer $8c2^n\ min = (8\cdot 2^{-40})2^n\ min = 2^{-37}\cdot 2^n\ min$. The time on the old machine will be 1 *min* if $2^{-37}\cdot 2^n = 1$, i.e., if $\boldsymbol{n = 37}$.