Solutions to CS / MCS 401 Exercise Set #4 (Fall 2007)

Exercise G In each part, we assume $n = 2^k$, so $k = \lg(n)$.

a)
$$\begin{aligned}
C(n) &= C(n/2) + 2n + 3 \\
&= \big(C(n/2^2) + 2(n/2) + 3\big) + 2n + 3 \\
&= C(n/2^2) + 2(n/2 + n) + 2\cdot 3 \\
&= \big(C(n/2^3) + 2(n/2^2) + 3\big) + 2(n/2 + n) + 2\cdot 3 \\
&= C(n/2^3) + 2(n/2^2 + n/2 + n) + 3\cdot 3 \\
&\;\;\vdots \\
&= C(n/2^k) + 2(n/2^{k-1} + \dots + n/2^2 + n/2 + n) + k\cdot 3 \\
&= C(1) + 2n(1/2^{k-1} + \dots + 1/2^2 + 1/2 + 1) + k\cdot 3 \\
&= 1 + 2n(2-1/2^{k-1}) + 3\lg(n) \\
&= 1 + 4n - 4 + 3\lg(n) \qquad \text{(since } n/2^{k-1} = 2^k/2^{k-1} = 2) \\
&= \mathbf{4n - 3 + 3\lg(n)}
\end{aligned}$$

b)
$$\begin{aligned}
C(n) &= 2C(n/2) + n\lg(n) \\
&= 2\big(2C(n/2^2) + n/2\cdot\lg(n/2)\big) + n\lg(n) \\
&= 2^2C(n/2^2) + n(\lg(n) - 1) + n\lg(n) \\
&= 2^2C(n/2^2) + 2n\lg(n) - n \\
&= 2^2\big(2C(n/2^3) + n/2^2\cdot\lg(n/2^2)\big) + 2n\lg(n) - n \\
&= 2^3C(n/2^3) + n(\lg(n) - 2) + 2n\lg(n) - n \\
&= 2^3C(n/2^3) + 3n\lg(n) - n(1+2) \\
&\;\;\vdots \\
&= 2^kC(n/2^k) + kn\lg(n) - n(1+2+\dots+k-1) \\
&= nC(1) + n(\lg(n))^2 - nk(k-1)/2 \\
&= n(\lg(n))^2 - n\lg(n)(\lg(n)-1)/2 \\
&= \mathbf{n(\lg(n))^2/2 + n(\lg(n))/2}
\end{aligned}$$

c)
$$\begin{aligned}
C(n) &= 4C(n/2) + 3n^2 && (*) \\
&= 4\big(4C(n/2^2) + 3(n/2)^2\big) + 3n^2 \\
&= 4^2C(n/2^2) + 4\cdot 3(n/2)^2 + n^2 \\
&= 4^2C(n/2^2) + 3(2n^2) && (**) \\
&= 4^2\big(4C(n/2^3) + 3(n/2^2)^2\big) + 3(2n^2) \\
&= 4^3C(n/2^3) + 4^2\cdot 3(n/2^2)^2 + 3(2n^2) \\
&= 4^3C(n/2^3) + 3(3n^2) && (***) \\
&\;\;\vdots
\end{aligned}$$

$= 4^k C(n/2^k) + 3(kn^2)$ following pattern on lines (*), (**), (***)

$= 4^k C(1) + 3\lg(n) n^2$

$= 4^k 0 + 3n^2 \lg(n) = \mathbf{3n^2 \lg(n)}$

Problem 4-4, parts (a), (c), (e), (h)

a) $T(n) = 3T(n/2) + n\lg(n)$.

In the Master Theorem, $a = 3$, $b = 2$, $E = \lg(3) \approx 1.59$, and $f(n) = n\lg(n)$. $f(n) = O(n^{E-\varepsilon})$, where we could take $\varepsilon = 0.1$. The Master Theorem (case 1) tells us $T(n) = \Theta(n^{\lg(3)}) \approx \mathbf{\Theta(n^{1.59})}$.

c) $T(n) = 4T(n/2) + n^2 sqrt(n) = 4T(n/2) + n^{5/2}$.

In the Master Theorem, $a = 4$, $b = 2$, $E = \lg(4) = 2$, $n^E = n^2$, and $f(n) = n^{5/2}$. $f(n) = \Omega(n^{E+\varepsilon})$, where we could take $\varepsilon = 0.1$. $af(n/b) = 4(n/2)^{5/2} = 2^{-1/2} n$, so $af(n/b) \le cf(n)$ where $c = 2^{-1/2} < 1$. The Master Theorem (case 3) tells us $T(n) = \Theta(f(n)) = \mathbf{\Theta(n^{5/2})}$.

e) $T(n) = 2T(n/2) + n/\lg(n)$.

In the notation of the Master Theorem, $a = 2$, $b = 2$, $E = \lg(2) = 1$, $f(n) = n/\lg(n)$, and $n^E = n$. The Master Theorem does not apply, as f(n) grows too rapidly for case (1) and not rapidly enough for case (2). But the extension to the Master Theorem in the handout tells us that T(n) = $\mathbf{\Theta(n \lg\lg(n))}$.

h) $T(n) = T(n-1) + \lg(n)$

$T(n) = T(n-1) + \lg(n)$

$= T(n-2) + \lg(n-1) + \lg(n)$

$= T(n-3) + \lg(n-2) + \lg(n-1) + \lg(n)$

$\vdots$

$= T(1) + \lg(2) + \lg(n-2) + \lg(n-1) + \lg(n)$

$= T(1) + \lg(n!)$

$= n\lg(n) - 1.44n + O(\lg(n))$

$= \mathbf{\Theta(n \lg(n)}$

Exercise H. Postponed to Week #5 exercises.