Rate of Growth of Functions

(The special case in which $\lim_{n\to\infty} f(n) / g(n)$ exists)

Let $g(n)$ be a fixed function.

$\omega(g(n))$

Functions $f(n)$ such that

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty.$$

Functions f(n) that dominate g(n), for large n.

$\Omega(g(n))$

$\Theta(g(n))$

Functions $f(n)$ such that

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c,$$

where $0 < c < \infty$.

Functions f(n) that are roughly proportional to g(n), for large n.

$O(g(n))$

$o(g(n))$

Functions $f(n)$ such that

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0.$$

Functions f(n) that are insignificant compared to g(n), for large n.

Rate of Growth of Functions

(The general case: $\lim_{n\to\infty} f(n) / g(n)$ need not exist)

Let $g(n)$ be a fixed function.

$\omega(g(n))$

Functions $f(n)$ such that

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$$

$\Omega(g(n))$

Functions $f(n)$ such that for all n sufficiently large,

$$\frac{f(n)}{g(n)} \geq c_1,$$

for some constant c_1 with $c_1 > 0$.

$\Omega(g(n)) \supseteq \omega(g(n)) \cup \Theta(g(n))$.

Note $g(n)/(1+\cos(n))$ is in $\Omega(g(n))$ (Let $c_1 = 0.5$), but not in $\omega(g(n)) \cup \Theta(g(n))$.

$O(g(n))$

Functions $f(n)$ such that for all n sufficiently large,

$$\frac{f(n)}{g(n)} \leq c_2,$$

for some constant c_2.

$O(g(n)) \supseteq o(g(n)) \cup \Theta(g(n))$.

Note $g(n)(1+\cos(n))$ is in $O(g(n))$ (Let $c_2 = 2$), but not in $o(g(n)) \cup \Theta(g(n))$.

$\Theta(g(n))$

Functions $f(n)$ such that for all n sufficiently large,

$$c_1 \leq \frac{f(n)}{g(n)} \leq c_2,$$

for some constants c_1 and c_2 with $0 < c_1 < c_2 < \infty$.

$o(g(n))$

Functions $f(n)$ such that

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0.$$

Note that, in the general case, some function $f(n)$ are in none of the categories above.

For example, if

$$f(n) = g(n)\, tan^2(n),$$

then $f(n)/g(n)$ takes on both values arbitrarily close to 0, and values arbitrarily large, as n increases.   This implies $\lim_{n\to\infty} f(n)/g(n)$ doesn't exist, and neither of the constants c_1 or c_2 exist.

Example: Here are various ways to write the approximation to lg(n!) given by Stirling's formula. Each line gives a more careful approximation than the line above it.

$$\lg(n!) = \Theta(n \lg(n))$$

$$\lg(n!) = n \lg(n) + o(n \lg(n))$$

$$\lg(n!) = n \lg(n) + \Theta(n)$$

$$\lg(n!) = n \lg(n) - \lg(e)\, n + o(n)$$

$$\lg(n!) = n \lg(n) - \lg(e)\, n + \Theta(\lg(n))$$

$$\lg(n!) = n \lg(n) - \lg(e)\, n + 0.5 \lg(n) + o(\lg(n))$$

$$\lg(n!) = n \lg(n) - \lg(e)\, n + 0.5 \lg(n) + \Theta(1)$$

` $$\lg(n!) = n \lg(n) - \lg(e)\, n + 0.5 \lg(n) + 0.5 \lg(2\pi) + \Theta(1/n)$$