Solutions to CS 401 / MCS 401 Quiz Number #1 — Summer 2007

1. a) By Stirling's formula, $n! \approx (n/e)^n \text{sqrt}(2\pi n)$. So

$$\lim_{n\to\infty} b^n n!/n^n = \lim_{n\to\infty} b^n (n/e)^n \text{sqrt}(2\pi n)/n^n = \lim_{n\to\infty} (b/e)^n \text{sqrt}(2\pi n)$$
$$= 0 \text{ if } b < e \text{ and } \infty \text{ if } b \geq e.$$

So $b^n n! \in O(n^n)$ when $b < e$.

b) $\lim_{n\to\infty} n^b 2^n/(n^2 3^n) = \lim_{n\to\infty} n^{b-2}(2/3)^n = 0$ (exponentials dominate polynomials). So there are no values of b for which $n^b 2^n \in \Omega(n^2 3^n)$.

c) $b^{\lg(n)} = n^{\lg(b)}$, so $\lim_{n\to\infty} b^{\lg(n)}/n^{\lg(5)} = \lim_{n\to\infty} n^{\lg(b)}/n^{\lg(5)} = \lim_{n\to\infty} n^{\lg(b)-\lg(5)}$.
$= 0$ if $b < 5$, 1 if $b = 5$, and ∞ if $b > 5$.

So $b^{\lg(n)} \in \Theta(n^{\lg(5)})$ exactly when $b = 5$.

2. a) $C(2) = C(1) + 2^2 + 1 = 2 + 4 + 1 = 7.$
$C(4) = C(2) + 4^2 + 1 = 7 + 16 + 1 = 24.$

b) Assume $n = 2^k$, so $k = \lg(n)$.

$$\begin{aligned}
C(n) &= C(n/2) + n^2 + 1 \\
&= C(n/2^2) + (n/2)^2 + 1 + n^2 + 1 \\
&= C(n/2^2) + \big((n/2)^2 + n^2\big) + 2 \\
&= C(n/2^3) + (n/2^2)^2 + 1 + \big((n/2)^2 + n^2\big) + 2 \\
&= C(n/2^3) + \big((n/2^2)^2 + (n/2)^2 + n^2\big) + 3 \\
&\vdots \\
&= C(n/2^k) + \big((n/2^{k-1})^2 + .. + (n/2^2)^2 + (n/2)^2 + n^2\big) + k \\
&= C(1) + n^2\big((1/4)^{k-1} + .. + (1/4)^2 + (1/4) + 1\big) + \lg(n) \\
&= 2 + n^2\big((1/4)^{k-1}(1/4) - 1\big) \,/\, ((1/4) - 1) + \lg(n) \\
&= 2 + (4/3)n^2(1 - 1/4^k) + \lg(n) \\
&= 2 + (4/3)n^2(1 - 1/n^2) + \lg(n) \qquad (\text{using } 4^k = 4^{\lg(n)} = n^{\lg(4)} = n^2) \\
&= 2/3 + (4/3)n^2 + \lg(n)
\end{aligned}$$