## Dynamic Programming: Example 2

 Longest Common SubsequenceProblem: Let $x_{1} x_{2} \ldots x_{m}$ and $y_{1} y_{2} \ldots y_{n}$ be two sequences over some alphabet. (We assume they are strings of characters.) Find a longest common subsequence (LCS) of $x_{1} x_{2} \ldots x_{m}$ and $y_{1} y_{2} \ldots y_{n}$.

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Example: }\quad\mp@subsup{x}{1}{}\mp@subsup{x}{2}{}\mp@subsup{x}{3}{}\mp@subsup{x}{4}{}\mp@subsup{x}{5}{}\mp@subsup{x}{6}{}\mp@subsup{x}{7}{}\mp@subsup{x}{8}{}=\underline{\mathbf{b}}\underline{\mathbf{c}}\underline{\mathbf{b}}\underline{\mathbf{f}}\mathbf{f}\underline{\mathbf{c}
    \mp@subsup{y}{1}{}}\mp@subsup{y}{2}{}\mp@subsup{y}{3}{}\mp@subsup{y}{4}{}\mp@subsup{y}{5}{\prime}\mp@subsup{y}{6}{\prime}\mp@subsup{y}{7}{}\mp@subsup{y}{8}{\prime}\mp@subsup{y}{9}{}=\mathbf{dab}\mathbf{e}\underline{\mathbf{a}}\underline{\mathbf{b}}\underline{\mathbf{f}}\underline{\mathbf{c}
    z z}\mp@subsup{z}{2}{}\mp@subsup{z}{3}{}\mp@subsup{z}{4}{}\mp@subsup{z}{5}{}\quad=\quad\mathbf{babfces is an LCS (shown below).
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Subproblems: Find an LCS of $x_{1} x_{2} \ldots x_{i}$ and $y_{1} y_{2} \ldots y_{j}(0 \leq i \leq m, 0 \leq j \leq m)$.

Optimal substructure: If $z=z_{1} z_{2} \ldots z_{p}$ is a LCS of $x_{1} x_{2} \ldots x_{m}$ and $y_{1} y_{2} \ldots y_{n}$, then at least one of these most hold.
i) $x_{m}=y_{n}$, and $z_{1} z_{2} \ldots z_{p-1}$ is an LCS of $x_{1} x_{2} \ldots x_{m-1}$ and $y_{1} y_{2} \ldots y_{n-1}$,
ii) $x_{m} \neq y_{n}$, and $z_{1} z_{2} \ldots z_{p}$ is an LCS of $x_{1} x_{2} \ldots x_{m-1}$ and $y_{1} y_{2} \ldots y_{n}$,
iii) $x_{m} \neq y_{n}$, and $z_{1} z_{2} \ldots z_{p}$ is an LCS of $x_{1} x_{2} \ldots x_{m}$ and $y_{1} y_{2} \ldots y_{n-1}$.

Let $c_{i j}=$ length of LCS of $x_{1} x_{2} \ldots x_{i}$ and $y=y_{1} y_{2} \ldots y_{j}$.

$$
\begin{aligned}
& c[i, j]= \begin{cases}0 & \text { if } i=0 \text { or } j=0, \\
1+c[i-1, j-1] & \text { if } x_{i}=y_{j}, \\
\max (c[i-1, j], c[i, j-1]) & \text { if } x_{i} \neq y_{j} .\end{cases} \\
& b[i, j]= \begin{cases}" \uparrow-" & \text { if } x_{i}=y_{j}, \\
" \uparrow " & \text { if } x_{i} \neq y_{j} \text { and } c[i-1, j] \geq c[i, j-1], \\
" \leftarrow " & \text { if } x_{i} \neq y_{j} \text { and } c[i-1, j]<c[i, j-1] .\end{cases}
\end{aligned}
$$

We compute the $c[i, j]$ and $b[i, j]$ in order of increasing $i+j$, or alternatively in order of increasing $i$, and for a fixed $i$, in order of increasing $j$.

$$
\text { Example: } \begin{array}{ll}
x_{1} x_{2} x_{3} x_{4} x_{5} x_{6} x_{7} x_{8}=\mathbf{b a c b f} \mathbf{f} \mathbf{b} \\
& y_{1} y_{2} y_{3} y_{4} y_{5} y_{6} y_{7} y_{8} y_{9}=\mathbf{d a b e a b f b c}
\end{array}
$$

Row $i$, column $j$ of the table below contains the value of $c[i, j]$ followed (except when $i=0$ or $j=0$ ) by that of $b[i, j]$

|  | 0 | $\begin{aligned} & \mathbf{1} \\ & \mathbf{d} \end{aligned}$ | $\begin{aligned} & \mathbf{2} \\ & \mathbf{a} \end{aligned}$ | $\begin{aligned} & \mathbf{3} \\ & \mathbf{b} \end{aligned}$ | $\begin{aligned} & 4 \\ & \mathrm{e} \end{aligned}$ | $\begin{aligned} & \mathbf{5} \\ & \mathbf{a} \end{aligned}$ | $\begin{aligned} & \mathbf{6} \\ & b \end{aligned}$ | $\begin{aligned} & 7 \\ & f \end{aligned}$ | $\begin{aligned} & \mathbf{8} \\ & \mathbf{b} \end{aligned}$ | 9 c |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 b | 0 | - | $\begin{aligned} & 0 \\ & \uparrow \end{aligned}$ | $1$ | $\begin{aligned} & 1 \\ & \leftarrow \end{aligned}$ | $\begin{aligned} & 1 \\ & \uparrow \end{aligned}$ | $1$ | $\begin{aligned} & 1 \\ & \leftarrow \end{aligned}$ | $\uparrow$ | $\begin{aligned} & 1 \\ & \leftarrow \end{aligned}$ |
| 2 a | 0 | ¢ | $1$ | $\uparrow$ | 1 | ${ }_{\uparrow}^{2}$ | $\begin{aligned} & 2 \\ & \leftarrow \end{aligned}$ | $\begin{aligned} & 2 \\ & \leftarrow \end{aligned}$ | $\begin{aligned} & 2 \\ & \leftarrow \end{aligned}$ | $\begin{aligned} & 2 \\ & \leftarrow \end{aligned}$ |
| 3 c | 0 | $\begin{aligned} & 0 \\ & \uparrow \end{aligned}$ | $\begin{aligned} & 1 \\ & \uparrow \end{aligned}$ | $\begin{aligned} & 1 \\ & \uparrow \end{aligned}$ | $\begin{aligned} & 1 \\ & \uparrow \end{aligned}$ | $\begin{aligned} & 2 \\ & \uparrow \end{aligned}$ | $\begin{aligned} & 2 \\ & \uparrow \end{aligned}$ | $\begin{aligned} & 2 \\ & \uparrow \end{aligned}$ | $\stackrel{2}{\uparrow}$ | $\stackrel{3}{4}$ |
| 4 b | 0 | ¢ | $\begin{aligned} & 1 \\ & \uparrow \end{aligned}$ | ${ }^{2}$ | $\stackrel{2}{\leftarrow}$ | 2 | ${ }^{3}$ | 3 $\leftarrow$ | ${ }^{3}$ | 3 $\uparrow$ |
| 5 f | 0 | $\begin{aligned} & 0 \\ & \uparrow \end{aligned}$ | $\begin{aligned} & 1 \\ & \uparrow \end{aligned}$ | $\begin{aligned} & 2 \\ & \uparrow \end{aligned}$ | $\begin{aligned} & 2 \\ & \uparrow \end{aligned}$ | $\begin{aligned} & 2 \\ & \uparrow \end{aligned}$ | $\begin{aligned} & 3 \\ & \uparrow \end{aligned}$ | $\begin{aligned} & 4 \\ & \uparrow \end{aligned}$ | $\begin{aligned} & 4 \\ & \leftarrow \end{aligned}$ | $\stackrel{4}{\leftarrow}$ |
| 6 f | 0 | $\begin{aligned} & 0 \\ & \uparrow \end{aligned}$ | $\begin{aligned} & 1 \\ & \uparrow \end{aligned}$ | $\begin{aligned} & 2 \\ & \uparrow \end{aligned}$ | $\begin{aligned} & 2 \\ & \uparrow \end{aligned}$ | $\begin{aligned} & 2 \\ & \uparrow \end{aligned}$ | $\begin{aligned} & 3 \\ & \uparrow \end{aligned}$ | $\begin{gathered} 4 \\ \uparrow \end{gathered}$ | $\begin{aligned} & 4 \\ & \uparrow \end{aligned}$ | $\uparrow$ |
| 7 c | 0 | $\begin{aligned} & 0 \\ & \uparrow \end{aligned}$ | $\begin{aligned} & 1 \\ & \uparrow \end{aligned}$ | $\begin{aligned} & 2 \\ & \uparrow \end{aligned}$ | $\begin{aligned} & 2 \\ & \uparrow \end{aligned}$ | $\begin{aligned} & 2 \\ & \uparrow \end{aligned}$ | $\begin{aligned} & 3 \\ & \uparrow \end{aligned}$ | $\begin{aligned} & 4 \\ & \uparrow \end{aligned}$ | $\begin{aligned} & 4 \\ & \uparrow \end{aligned}$ | ${ }_{\uparrow}^{5}$ |
| 8 b | 0 | $\begin{aligned} & 0 \\ & \uparrow \end{aligned}$ | $\begin{aligned} & 1 \\ & \uparrow \end{aligned}$ | ${ }_{2}^{2}$ | 2 | $\begin{aligned} & 2 \\ & \uparrow \end{aligned}$ | $\begin{aligned} & 3 \\ & \uparrow \end{aligned}$ | $\begin{aligned} & 4 \\ & \uparrow \end{aligned}$ | ${ }_{\sim}^{5}$ | 5 $\uparrow$ |

We can compute each table element in constant time, so the entire table takes $\Theta(m n)$ time.

We can write down an LCS by starting in the lower right corner and following the arrows backward.

Whenever we reach a square containing a " $\uparrow$ ", say in row $i$ and column $j$, we insert the character $x_{i}=y_{j}$ at the beginning of the subsequence.

|  | 0 | $\begin{aligned} & \mathbf{1} \\ & \mathbf{d} \end{aligned}$ | $\mathbf{a}$ | $\begin{aligned} & \hline \mathbf{3} \\ & \mathbf{b} \end{aligned}$ | $\begin{aligned} & 4 \\ & \mathrm{e} \\ & \hline \end{aligned}$ | $\begin{aligned} & \mathbf{5} \\ & \mathbf{a} \end{aligned}$ | b | $\begin{aligned} & 7 \\ & \mathrm{f} \end{aligned}$ | $\begin{aligned} & 8 \\ & \mathbf{b} \end{aligned}$ | $\begin{aligned} & 9 \\ & \mathbf{c} \end{aligned}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 b | 0 | $\begin{aligned} & 0 \\ & \uparrow \end{aligned}$ | $\begin{aligned} & 0 \\ & \uparrow \end{aligned}$ | $\begin{aligned} & \mathbf{1} \\ & \uparrow \end{aligned}$ | $\begin{aligned} & \mathbf{1} \\ & \leftarrow \end{aligned}$ | $\begin{aligned} & 1 \\ & \uparrow \end{aligned}$ | $\hat{\imath}^{1}$ | $\begin{aligned} & 1 \\ & \leftarrow \end{aligned}$ | $\hat{\Lambda}^{1}$ | $\begin{aligned} & 1 \\ & \leftarrow \end{aligned}$ |
| 2 a | 0 | $\begin{aligned} & 0 \\ & \uparrow \end{aligned}$ | $1$ | $\begin{aligned} & 1 \\ & \uparrow \end{aligned}$ | $\begin{aligned} & 1 \\ & \uparrow \end{aligned}$ | ${ }_{\uparrow}^{2}$ | $\begin{aligned} & 2 \\ & \leftarrow \end{aligned}$ | $\begin{aligned} & 2 \\ & \leftarrow \end{aligned}$ | $\begin{aligned} & 2 \\ & \leftarrow \end{aligned}$ | $\begin{aligned} & 2 \\ & \leftarrow \end{aligned}$ |
| 3 c | 0 | $\begin{aligned} & 0 \\ & \uparrow \end{aligned}$ | $\stackrel{1}{\uparrow}$ | $\stackrel{1}{\uparrow}$ | $\stackrel{1}{\uparrow}$ | $\begin{aligned} & \mathbf{2} \\ & \uparrow \end{aligned}$ | $\underset{\uparrow}{2}$ | $\begin{aligned} & 2 \\ & \uparrow \end{aligned}$ | $\stackrel{2}{\uparrow}$ | ${ }_{4}^{3}$ |
| 4 b | 0 | $\begin{aligned} & 0 \\ & \uparrow \end{aligned}$ | $\stackrel{1}{\uparrow}$ | ${ }_{\uparrow}^{2}$ | $\begin{aligned} & 2 \\ & \leftarrow \end{aligned}$ | $\begin{aligned} & 2 \\ & \uparrow \end{aligned}$ | $3$ | $\begin{aligned} & 3 \\ & \leftarrow \end{aligned}$ | ${ }_{4}^{3}$ | $\begin{aligned} & 3 \\ & \uparrow \end{aligned}$ |
| 5 f | 0 | $\begin{aligned} & 0 \\ & \uparrow \end{aligned}$ | $\begin{aligned} & 1 \\ & \uparrow \end{aligned}$ | $\begin{aligned} & 2 \\ & \uparrow \end{aligned}$ | $\stackrel{2}{\uparrow}$ | $\underset{\uparrow}{2}$ | $\begin{aligned} & 3 \\ & \uparrow \end{aligned}$ | ${ }_{\uparrow}^{4}$ | $\stackrel{4}{\leftarrow}$ | $\stackrel{4}{\leftarrow}$ |
| 6 f | 0 | $\begin{aligned} & 0 \\ & \uparrow \end{aligned}$ | $\stackrel{1}{\uparrow}$ | $\begin{aligned} & 2 \\ & \uparrow \end{aligned}$ | $\begin{aligned} & 2 \\ & \uparrow \end{aligned}$ | $\stackrel{2}{\uparrow}$ | 3 $\uparrow$ | $\stackrel{4}{\wedge}$ | $\stackrel{4}{4}$ | $\stackrel{4}{\uparrow}$ |
| 7 c | 0 | $\begin{aligned} & 0 \\ & \uparrow \end{aligned}$ | $\stackrel{1}{\uparrow}$ | $\begin{aligned} & 2 \\ & \uparrow \end{aligned}$ | $\stackrel{2}{\uparrow}$ | $\stackrel{2}{\uparrow}$ | $\begin{aligned} & 3 \\ & \uparrow \end{aligned}$ | $\begin{aligned} & 4 \\ & \uparrow \end{aligned}$ | $\stackrel{4}{4}$ | $5$ |
| 8 b | 0 | $\begin{aligned} & 0 \\ & \uparrow \end{aligned}$ | $\stackrel{1}{\uparrow}$ | ${ }_{\uparrow}^{2}$ | $\underset{\uparrow}{2}$ | $\underset{\uparrow}{2}$ | $\begin{aligned} & 3 \\ & \uparrow \end{aligned}$ | $\begin{aligned} & 4 \\ & \uparrow \end{aligned}$ | $5$ | 5 $\uparrow$ |

An LCS is $x_{1} x_{2} x_{4} x_{5} x_{7}=y_{3} y_{5} y_{6} y_{7} y_{9}=$ babfc.

This computation takes $\Theta(m+n)$ time.

