Dynamic Programming: Example 1
Order of Matrix Multiplication

Problem: What is the optimal order for computing $M_1 \times M_2 \times \ldots \times M_n$, where M_i is an r_i by c_i matrix.

Optimal substructure: If the optimal order for $M_1 \times M_2 \times \ldots \times M_n$ is of the form

$$(M_1 \times \ldots \times M_k) \times (M_{k+1} \times \ldots \times M_n),$$

then $M_1 \times \ldots \times M_k$ is computed by optimal order, and $M_{k+1} \times \ldots \times M_n$ is computed by optimal order.

m_{ij} = number of scalar multiplications used in computing $M_i \times M_{i+1} \times \ldots \times M_j$ using optimal order ($1 \le i \le j \le n$).

$$(*) \qquad m_{ij} = \begin{cases} 0 & \text{if } i = j, \\ \min_{i \le k \le j-1}(m_{ik} + m_{k+1,j} + r_i c_k c_j) & \text{if } i < j. \end{cases}$$

v_{ij} = value of k giving minimum above, if $i < j$.

We start with $m_{ii} = 0$ for $i = 1,2,\ldots n$, and compute and save all the m_{ij} and v_{ij} in order of increasing $j–i$.

That way, when we compute m_{ij}, we will have already computed all the m_{ik} and $m_{k+1,j}$.

When we are done,

m_{1n} = number of scalar multiplications required to compute $M_1 \times M_2 \times \ldots \times M_n$ using the optimal order.

$V = (v_{ij})$ contains the information needed to determine the optimal order in $O(n)$ time.

Example: Compute $M_1 \times M_2 \times M_3 \times M_4 \times M_5 \times M_6$ using optimal order, where M_i is an r_i by c_i matrix.

i	r_i	c_i
1	5	20
2	20	10
3	10	7
4	7	60
5	60	8
6	8	40

We compute

i	j	m_{ij}	v_{ij}
1	1	0	
2	2	0	
3	3	0	
4	4	0	
5	5	0	
6	6	0	
1	2	1000	1
2	3	1400	2
3	4	4200	3
4	5	3360	4
5	6	19200	5
1	3	1350	2
2	4	9800	3
3	5	3920	3
4	6	5600	5
1	4	3450	3
2	5	5520	2
3	6	7120	5
1	5	4990	3
2	6	11920	5
1	6	6590	5

We compute a typical entry as follows:

$$
\begin{aligned}
m_{25} &= \min(\, m_{22}+m_{35}+r_2c_2c_5,\ m_{23}+m_{45}+r_2c_3c_5,\ m_{24}+m_{55}+r_2c_4c_5) \\
&= \min(\, 0+3920+20\cdot 10\cdot 8,\ 1400+3360+20\cdot 7\cdot 8, \\
&\qquad 9800+0+20\cdot 60\cdot 8) \\
&= \min(\, \underbrace{5520}_{k\,=\,2},\ 5880,\ 19400) \\
&= 5520 \\
v_{25} &= 2
\end{aligned}
$$

So the number of scalar multiplications is 6590. We determine the optimal order by

$v_{16} = 5$, so optimal order is $(M_1 \times M_2 \times M_3 \times M_4 \times M_5) \times M_6$.

$v_{15} = 3$, so optimal order is $((M_1 \times M_2 \times M_3) \times (M_4 \times M_5)) \times M_6$.

$v_{13} = 2$, so optimal order is $(((M_1 \times M_2) \times M_3) \times (M_4 \times M_5)) \times M_6$.