Rate of Growth of Functions

(The special case in which $\lim_{n\to\infty} f(n) / g(n)$ exists)

Let $g(n)$ be a fixed function.

$\Omega(g(n))$

$\omega(g(n))$

Functions $f(n)$ such that

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty.$$

Functions $f(n)$ that dominate $g(n)$, for large n.

$\Theta(g(n))$

Functions $f(n)$ such that

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c,$$

where $0 < c < \infty$.

Functions $f(n)$ that are roughly proportional to $g(n)$, for large n.

$O(g(n))$

$o(g(n))$

Functions $f(n)$ such that

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0.$$

Functions $f(n)$ that are insignificant compared to $g(n)$, for large n.

Rate of Growth of Functions

(The general case: $\lim_{n\to\infty} f(n) / g(n)$ need not exist)

Let $g(n)$ be a fixed function.

$\omega(g(n))$

Functions $f(n)$ such that

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$$

$\Omega(g(n))$

Functions $f(n)$ such that for all n sufficiently large,

$$\frac{f(n)}{g(n)} \geq c_1,$$

for some constant c_1 with $c_1 > 0$.

$\Omega(g(n)) \supseteq \omega(g(n)) \cup \Theta(g(n))$.

Note $g(n)/(1+\cos(n))$ is in $\Omega(g(n))$ (Let $c_1 = 0.5$), but not in $\omega(g(n)) \cup \Theta(g(n))$.

$O(g(n))$

Functions $f(n)$ such that for all n sufficiently large,

$$\frac{f(n)}{g(n)} \leq c_2,$$

for some constant c_2.

$O(g(n)) \supseteq o(g(n)) \cup \Theta(g(n))$.

Note $g(n)(1+\cos(n))$ is in $O(g(n))$ (Let $c_2 = 2$), but not in $o(g(n)) \cup \Theta(g(n))$.

$\Theta(g(n))$

Functions $f(n)$ such that for all n sufficiently large,

$$c_1 \leq \frac{f(n)}{g(n)} \leq c_2,$$

for some constants c_1 and c_2 with $0 < c_1 < c_2 < \infty$.

$o(g(n))$

Functions $f(n)$ such that

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0.$$

Note that, in the general case, some function $f(n)$ are in none of the categories above.

For example, if

$$f(n) = g(n)\, tan^2(n),$$

then $f(n)/g(n)$ takes on both values arbitrarily close to 0, and values arbitrarily large, as n increases. This implies $\lim_{n\to\infty} f(n)/g(n)$ doesn't exist, and neither of the constants c_1 or c_2 exist.

Example: Here are various ways to write the approximation to $\lg(n!)$ given by Stirling's formula. Each line gives a more careful approximation than the line above it.

$$\lg(n!) = \Theta(n \lg(n))$$

$$\lg(n!) = n \lg(n) + o(n \lg(n))$$

$$\lg(n!) = n \lg(n) + \Theta(n)$$

$$\lg(n!) = n \lg(n) - \lg(e)\, n + o(n)$$

$$\lg(n!) = n \lg(n) - \lg(e)\, n + \Theta(\lg(n))$$

$$\lg(n!) = n \lg(n) - \lg(e)\, n + 0.5 \lg(n) + o(\lg(n))$$

$$\lg(n!) = n \lg(n) - \lg(e)\, n + 0.5 \lg(n) + \Theta(1)$$

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$$\lg(n!) = n \lg(n) - \lg(e)\, n + 0.5 \lg(n) + 0.5 \lg(2\pi) + \Theta(1/n)$$

Rate of Growth of Functions: An Example

Let $\boldsymbol{g(n) = n^2 \lg(n)}$.

$f(n)$	$o(g(n))$	$\Omega(g(n))$		$f(n)/g(n)$ as $n \to \infty$
		$\Theta(g(n))$	$\omega(g(n))$	
	$O(g(n))$			
$8n^2 \lg(\lg(n))$	✓			$\to 0$
$3n^2 \lg(n) + 12n^2 - 56n$		✓		$\to 3$
$n^2 \lg(n^5)$		✓		$= 5$
$n^2 \ln(7n)$		✓		$\to \ln(2)$
$n^3 + 3n^2$			✓	$\to \infty$
$(6n^5+n^4)\lg(n)/(n^3-7n^2)$		✓		$\to 6$
$n^2 \lg(n)^3$			✓	$\to \infty$
$n^{2.1}/\lg(n)^3$			✓	$\to \infty$
$5n^2 \lg(n)/\lg(\lg(n))$	✓			$\to 0$
$4^{\lg(n)} 2^{\lg(\lg(n))}$		✓		$= 1$
$n^{1.9} \lg(n)^5$	✓			$\to 0$
$\lg(n)^{\lg(n)}$			✓	$\to \infty$
$\lg(n!)$	✓			$\to 0$
$\lg((n!)^n)$		✓		$\to 1$
$n^2 \lg(n)(2+\sin(n))$		✓		oscillates in (1,3)
$n^2 \lg(n)(1+\sin(n))$	in $O(g(n))$ only			oscillates in (0,2)
$n^2 \lg(n) \tan^2(n)$	in none of above			oscillates in $(0,\infty)$