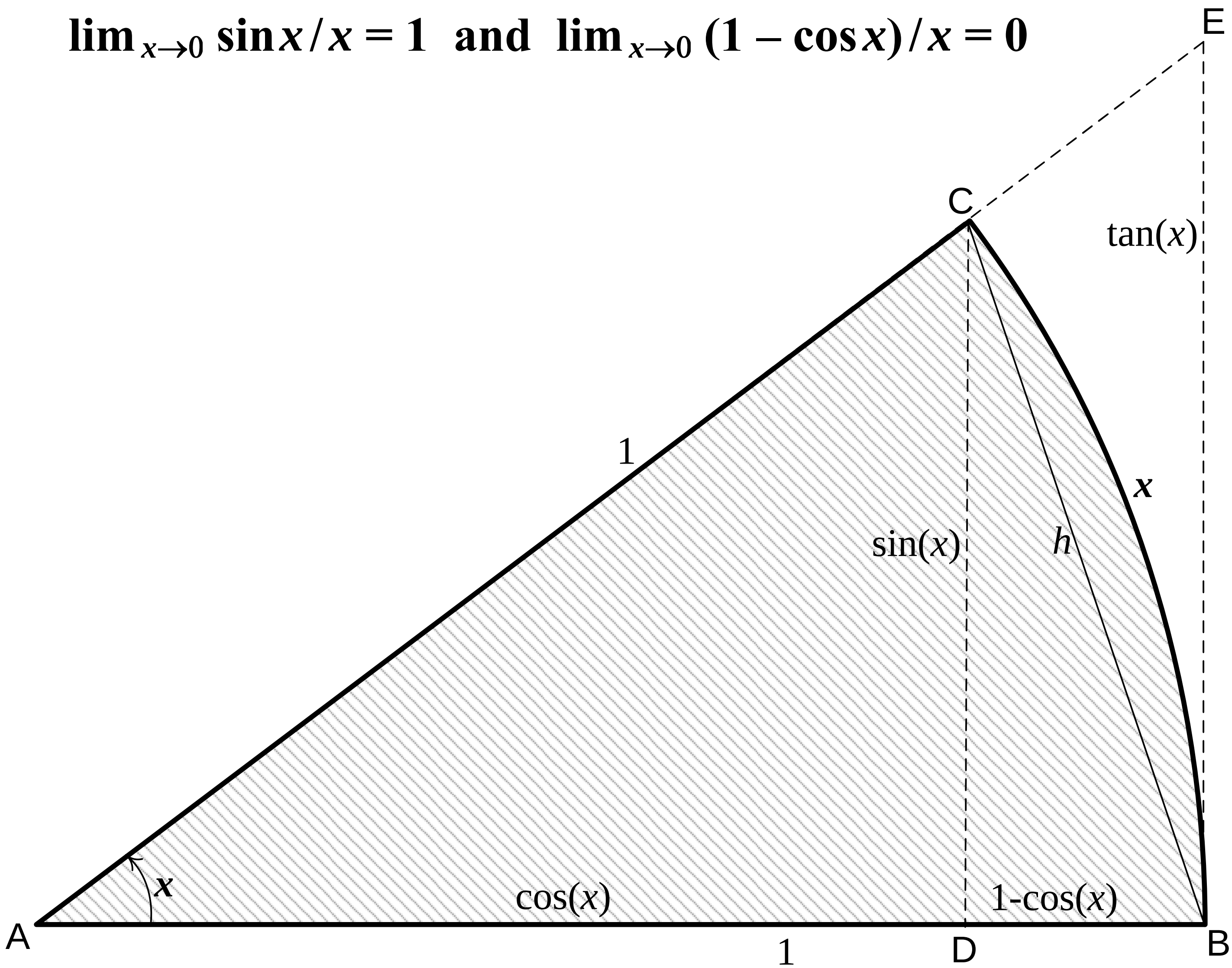
$\lim_{x\to 0} \sin x / x = 1$ and $\lim_{x\to 0} (1 - \cos x)/x = 0$



Let x be an angle with $0 < x < \pi/2$. In right triangle CDB, the Theorem of Pythagoras tells us that

$$h^2 = \sin^2 x + (1 - \cos x)^2 = \sin^2 x + 1 - 2\cos x + \cos^2 x = 2 - 2\cos x.$$

Note $h \le x$ (A straight line is shortest path between two points), so $h^2 \le x^2$, and hence $2 - 2\cos x \le x^2$. Thus

$$\mathbf{1 - \cos x \le x^2/2}, \quad \textbf{or} \qquad \textbf{(1)}$$

$$\mathbf{\cos x \ge 1 - x^2/2.} \qquad \textbf{(2)}.$$

Although we assumed that x is small and positive, these inequalities are equally valid for negative x, since they involve only even functions.

Dividing (1) by x, and using the fact that $\cos x \le 1$, we obtain

$$\mathbf{0 \le (1 - \cos x)/x \le x/2} \qquad \textbf{if } \mathbf{x > 0,}$$

$$\mathbf{0 \ge (1 - \cos x)/x \ge x/2} \qquad \textbf{if } \mathbf{x < 0.}$$

These combine to give

$$\mathbf{|(1 - \cos x)/x| \le |x|/2 \qquad \text{if } x \ne 0.}$$

Since $\lim_{x\to 0} |x|/2 = 0$, $\lim_{x\to 0} (1 - \cos x)/x = 0$.

Again assume x is positive, and note

$$\text{(Triangle ABC)} \subseteq \text{(shaded area)} \subseteq \text{(Triangle ABE).} \qquad \mathbf{(3)}$$

Triangle ABC has base 1 and height $\sin x$. Area = $1(\sin x)/2 = (\sin x)/2$.
The shaded area is $x/2\pi$ of the unit circle. Area = $(x/2\pi)(\pi 1^2) = x/2$.
Right triangle ABE has base 1 and height $\tan x$. Area = $(\tan x)/2$.

So $(\sin x)/2 \le x/2 \le (\tan x)/2$, and $\sin x \le x \le \tan x$.

From $\sin x \le x$, we obtain $\mathbf{\sin x / x \le 1}$. From $x \le \tan x = \sin x/\cos x$, we multiply by $(\cos x)/x$ to obtain $\cos x \le \sin x/x$. In view of (2), we can substitute $1 - x^2/2$ to obtain

$$\mathbf{1 - x^2/2 \le \sin x/x \le 1} \qquad \mathbf{(4)}$$

We assumed $0 < x < \pi/2$. But since $1 - x^2/2$ and $\sin x/x$ are even functions, (4) holds when $0 > x > -\pi/2$ as well.

Let $f(x) = 1 - x^2/2$ and $g(x) = 1$. Then

$$f(x) \le \sin x/x \le g(x) \text{ and } \lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 1,$$

$\lim_{x\to 0} \sin x/x = 1$.