Examples of Syndrome Decoding

Ex 1 Let C_1 be linear binary [6,3,3] code with generator matrix

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

and parity check matrix

$$H = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The syndromes and coset leaders are:

Syndrome	Coset Leader(s)
000	$\mathbf{0}$
001	$\boldsymbol{e}_6$
010	$\boldsymbol{e}_5$
011	$\boldsymbol{e}_1$
100	$\boldsymbol{e}_4$
101	$\boldsymbol{e}_2$
110	$\boldsymbol{e}_3$
111	$\boldsymbol{e}_1+\boldsymbol{e}_4$, $\boldsymbol{e}_2+\boldsymbol{e}_5$, $\boldsymbol{e}_3+\boldsymbol{e}_6$

$\mathbf{0} = (0,0,0,0,0,0)$,
$\boldsymbol{e}_1 = (1,0,0,0,0,0)$,

$\boldsymbol{e}_6 = (0,0,0,0,0,1)$,

Say we receive the vector $\boldsymbol{v} = (1\,1\,1\,1\,0\,1)$.

We know $\boldsymbol{v} = \boldsymbol{c} + \boldsymbol{e}$, where **c** is the codeword transmitted, and $\boldsymbol{e}$ is the error vector.

$$\boldsymbol{e}\mathbf{H}^T = (\boldsymbol{v} - \boldsymbol{c})\mathbf{H}^T = \boldsymbol{v}\mathbf{H}^T - \boldsymbol{c}\mathbf{H}^T = \boldsymbol{v}\mathbf{H}^T - \mathbf{0} = \boldsymbol{v}\mathbf{H}^T = (101).$$

The table tells us that $\boldsymbol{e}$ (and $\boldsymbol{v}$) are in the coset with leader $\boldsymbol{e}_2$. Under nearest-neighbor decoding, we want $wt(\boldsymbol{e})$ to be as small as possible, so we assume $\boldsymbol{e} = \boldsymbol{e}_2$.

So $\boldsymbol{c} = \boldsymbol{v} - \boldsymbol{e}_2 = (1\,1\,1\,1\,0\,1) - (0\,1\,0\,0\,0\,0) =$ **(101101)**.

The original message was simply the information symbols in $\boldsymbol{c}$ (the first three positions), or **101**.

Say we receive the vector $\boldsymbol{v} = \boldsymbol{c} + \boldsymbol{e} = (1\,0\,0\,1\,0\,0)$.

We compute $\boldsymbol{v}\mathbf{H}^T = (111)$, and $\boldsymbol{e}$ is in the coset of $\boldsymbol{e}_1+\boldsymbol{e}_4$. But there are three equally likely alternatives for the error vector. We have detected errors (probably two errors), but we cannot correct them.

Note we can always *correct one error* in a block, and occasionally we can *detect two errors*.

There are C(6,2) = 15 ways in which two errors can occur.

For 3 of these ways, and can detect (but not correct) the errors.

For the other 12, we compute $\boldsymbol{e}$ and $\boldsymbol{c}$ incorrectly (although in 3 of them, only the parity-check positions are affected).

Ex 2 Let C_2 be linear binary [7,2,4] code with generator matrix

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

and parity check matrix

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The syndromes and coset leaders are

Syndrome	Coset Leader(s)
0 0 0 0 0	$\mathbf{0}$
0 0 0 0 1	e_7
0 0 0 1 0	e_6
0 0 0 1 1	e_1+e_2, e_6+e_7
0 0 1 0 0	e_5
0 0 1 0 1	e_5+e_7
0 0 1 1 0	e_5+e_6
0 0 1 1 1	$e_1+e_2+e_5$, $e_5+e_6+e_7$
0 1 0 0 0	e_4
0 1 0 0 1	e_4+e_7
0 1 0 1 0	e_4+e_6
0 1 0 1 1	$e_1+e_2+e_4$, $e_4+e_6+e_7$
0 1 1 0 0	e_4+e_5
0 1 1 0 1	e_1+e_3
0 1 1 1 0	e_2+e_3
0 1 1 1 1	$e_1+e_3+e_6$, $e_2+e_3+e_7$
1 0 0 0 0	e_3
1 0 0 0 1	e_3+e_7
1 0 0 1 0	e_3+e_6
1 0 0 1 1	$e_1+e_2+e_3$, $e_3+e_6+e_7$
1 0 1 0 0	e_3+e_5
1 0 1 0 1	e_1+e_4
1 0 1 1 0	e_2+e_4
1 0 1 1 1	$e_1+e_4+e_6$, $e_2+e_4+e_7$
1 1 0 0 0	e_3+e_4
1 1 0 0 1	e_1+e_5
1 1 0 1 0	e_2+e_5
1 1 0 1 1	$e_1+e_5+e_6$, $e_2+e_5+e_7$
1 1 1 0 0	e_1+e_7, e_2+e_6
1 1 1 0 1	e_1
1 1 1 1 0	e_2
1 1 1 1 1	e_1+e_6, e_2+e_7

Say we receive the vector $\boldsymbol{v} = \boldsymbol{c} + \boldsymbol{e} = (1\,1\,1\,0\,1\,1\,0)$.

We compute $\boldsymbol{v}\mathbf{H}^{\mathrm{T}} = (1\,0\,1\,0\,1)$, and $\boldsymbol{e}$ is in the coset of $\boldsymbol{e}_1+\boldsymbol{e}_4$. Since $\boldsymbol{e}_1+\boldsymbol{e}_4$ is the unique coset leader, we assume $\boldsymbol{e} = \boldsymbol{e}_1+\boldsymbol{e}_4$, and compute $\boldsymbol{c} = \boldsymbol{v} - \boldsymbol{e}_2 = (1\,1\,1\,0\,1\,1\,0) - (1\,0\,0\,1\,0\,0\,0) = \mathbf{(0\,1\,1\,1\,1\,1\,0)}$. We then decode to the information symbols (first two positions), obtaining **0 1**.

Code C_2 can

i) Always correct one error in a 7-bit encoded block,

ii) Always detect two errors in a 7-bit encoded block, and usually correct them. There are C(7,2) = 21 ways in which two errors can occur. Of these, 15 can be corrected, and the other 6 only detected.

iii) Sometimes detect (but not correct) three errors in a 7-bit block. There are C(7,3) = 35 ways in which 3 errors can occur. Of these, 12 can be detected; the remaining 23 cause us to determine the error vector $\boldsymbol{e}$ and codeword $\boldsymbol{c}$ incorrectly (although in 7 of the 23, only the parity-check positions are affected).