Basics of Complex Numbers

A complex number is a formal expression $x+iy$, where x and y are real numbers, i is a formal object which satisfies $i\cdot i=-1=-1+i0$. The *real* part of $z=x+iy$, denoted $\Re z$, is x; the *imaginary* part of $z=x+iy$, denoted $\Im z$, is y.

- **Addition:**

If $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$, the *sum* is $z_1+z_2=(x_1+x_2)+i(y_1+y_2)$. Thus

$$\begin{aligned}\Re(z_1+z_2)&=\Re z_1+\Re z_2,\\ \Im(z_1+z_2)&=\Im z_1+\Im z_2.\end{aligned}$$

- Multiplication:

If $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$, the *product* z_1z_2 is

$$\begin{aligned}z_1\cdot z_2&=(x_1+iy_1)\cdot(x_2+iy_2)\\ &=(x_1x_2-y_1y_2)+i(y_1x_2+x_1y_2).\end{aligned}$$

Here we use the relation $i\cdot i=i^2=-1$. We also write $i=\sqrt{-1}$.

- **Complex Numbers, Points, and Vectors**

The complex number $z=x+iy$ can be identified with a point in the x–y coordinate plane P with coordinates (x,y). another useful view is to identify the point $P(x,y)$ [complex number $x+iy$] with the *vector* or arrow $\overrightarrow{OP}$ from the origin to P [z].

Addition of complex numbers is best understood in terms of addition of vectors: The point [vector] corresponding to z_2 added to z_1 is the *point* z_1 shifted by the *vector* z_2.

- **Modulus and Conjugate**

The *modulus* or *absolute value* of a complex number $z=x+iy$ is defined as

$$|z|=\sqrt{x^2+y^2}$$

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The conjugate of a complex number $z=x+iy$ is defined as

$$\bar{z}=\overline{x+iy}=x-iy.$$

Note that

$$z\bar{z}=|z|^2=x^2+y^2$$

and

$$\begin{aligned}\Re z&=\frac{1}{2}\left(z+\bar{z}\right),\\ \Im z&=\frac{1}{2i}\left(z-\bar{z}\right).\end{aligned}$$

- **Polar Coordinates**

In the plane, a point (x,y) [complex number $z=x+iy$] (not O) is completely determined by its distance from the origin $r=|z|=\sqrt{x^2+y^2}$ and the angle θ from the positive x–axis to the ray Oz from the origin to z.

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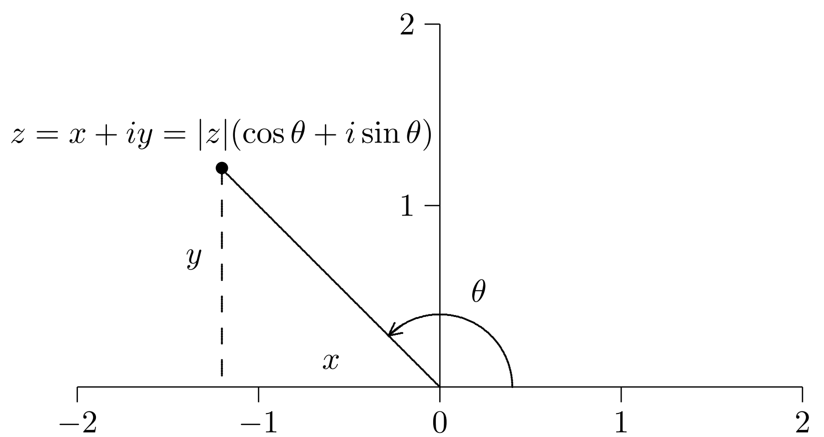

The pair (r, θ) are polar coordinates of the point $P(x, y)$ or the complex number $z = x+iy$. We have:

$$
\begin{aligned}
x &= r\cos\theta,\\
y &= r\sin\theta,\\
z &= r\left(\cos\theta + i\sin\theta\right),\\
z &= |z|\left(\cos\theta + i\sin\theta\right).
\end{aligned}
$$

Note that r is the *modulus* of z. The angle θ is called an *argument* of z, written $\arg(z)$. For convenience we introduce the notation

$$\text{cis}\theta = \cos\theta + i\sin\theta$$

so that for $z \neq 0$,

$$z = |z|\,\text{cis}(\arg(z)).$$

It is important that $\arg(z)$ is not uniquely determined. If θ is an argument of z, then θ + any integer multiple of 2π is also an argument of z.[1]

Note that

$$\arg(\bar{z}) = -\arg(z).$$

- **Multiplication and Polar Coordinates**

Geometrically, multiplication by a nonzero z is best understood in terms of polar coordinates.

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[1] A particular choice for $\arg(z)$, for example, the unique $\arg(z)$ that satisfies $0 \leq \arg(z) < 2\pi$ is written $\text{Arg}(z)$.

If $z_1 = |z_1| \operatorname{cis}(\theta_1)$, $z_2 = |z_2| \operatorname{cis}\theta_2$, verify that

$$z_1 \cdot z_2 = |z_1| \, |z_2| \operatorname{cis}(\theta_1 + \theta_2) .$$

Thus:

- The modulus of the product = the product of the moduli.

$$|z_1 z_2| = |z_1| \, |z_2| .$$

- An argument of the product = the sum of the arguments.[2]

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) .$$

- **Reciprocal**

  If $z \neq 0$, the reciprocal of z, $\frac{1}{z}$, can be calculated as

$$\begin{aligned} \frac{1}{z} &= \frac{\bar{z}}{z\bar{z}} \\ &= \frac{\bar{z}}{|z|^2} \\ &= \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2} \\ &= \frac{1}{|z|} \operatorname{cis}(-\arg(z)). \end{aligned}$$

[2] Give an example of complex numbers z_1 and z_2 such that

$$\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) - 2\pi.$$