Laurent Series

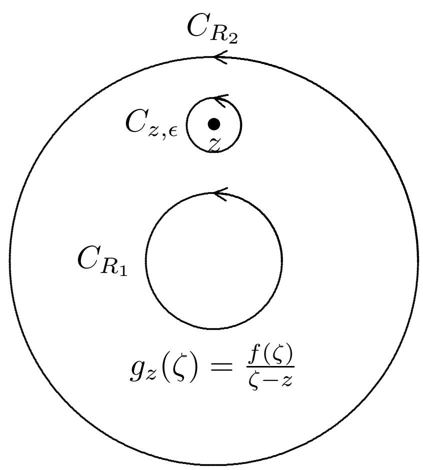
Theorem. *Let $f(z)$ be analytic in the closed region*

$$D_{R_1,R_2} = \{0 < R_1 \leq |z| \leq R_2\}.$$

Then for $R_1 < |z| < R_2$,

$$f(z) = \frac{1}{2\pi i}\oint_{C_{R_2}} \frac{f(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i}\oint_{C_{R_1}} \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

Proof: The proof is similar in spirit to the proof of the Cauchy Integral Formula.
Fix z. For ϵ small, let $C_{z,\epsilon} = \{\zeta\,||\zeta - z| = \epsilon\}$ is in between C_{R_1} and C_{R_2}. Define $g_z(\zeta) = \dfrac{f(\zeta)}{\zeta - z}$. Then $g_z(\zeta)$ is an analytic function of ζ in the region between the two circles C_{R_1} and C_{R_2} and outside $C_{z,\epsilon}$.



$$\begin{aligned} 2\pi i\, f(z) &= \oint_{C_{z,\epsilon}} g_z(\zeta)\,d\zeta \\ &= \oint_{C_{R_2}} g_z(\zeta)\,d\zeta - \oint_{C_{R_1}} g_z(\zeta)\,d\zeta \\ &= \oint_{C_{R_2}} \frac{f(\zeta)}{\zeta - z}\,d\zeta - \oint_{C_{R_1}} \frac{f(\zeta)}{\zeta - z}\,d\zeta. \end{aligned}$$

Next we write

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint_{C_{R_2}} \frac{f(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i}\oint_{C_{R_1}} \frac{f(\zeta)}{\zeta - z}\,d\zeta \\ &= f_1(z) + f_2(z). \end{aligned}$$

Proceeding as before

$$\begin{aligned}
f_1(z) &= \frac{1}{2\pi i}\oint_{C_{R_2}} \frac{f(\zeta)}{\zeta - z}\, d\zeta. \\
&= \frac{1}{2\pi i}\oint_{C_{R_2}} f(\zeta)\frac{1}{\zeta}\sum_{n=0}^{\infty}\left(\frac{z}{\zeta}\right)^n d\zeta \\
&= \frac{1}{2\pi i}\oint_{C_{R_2}} f(\zeta)\frac{1}{\zeta}\sum_{n=0}^{N}\left(\frac{z}{\zeta}\right)^n d\zeta \\
&+ \frac{1}{2\pi i}\oint_{C_{R_2}} f(\zeta)\frac{1}{\zeta}\frac{\left(\frac{z}{\zeta}\right)^{N+1}}{1-\frac{z}{\zeta}}\, d\zeta \\
&= \sum_{n=0}^{\infty} a_n z^n,
\end{aligned}$$

where

$$a_n = \frac{1}{2\pi i}\oint_{C_{R_2}} f(\zeta)\frac{1}{\zeta^{n+1}}\, d\zeta.$$

Note that $f_1(z)$ is analytic in the region $\{z\,||z| \le R_2\}$.

In the same spirit,

$$\begin{aligned}
f_2(z) &= -\frac{1}{2\pi i}\oint_{C_{R_1}} \frac{f(\zeta)}{\zeta - z}\, d\zeta \\
&= \frac{1}{2\pi i}\oint_{C_{R_1}} f(\zeta)\frac{1}{z}\frac{1}{1-\dfrac{\zeta}{z}}\, d\zeta \\
&= \frac{1}{2\pi i}\oint_{C_{R_1}} f(\zeta)\frac{1}{z}\sum_{m=0}^{\infty}\left(\frac{\zeta}{z}\right)^m d\zeta \\
&= \frac{1}{2\pi i}\oint_{C_{R_1}} f(\zeta)\frac{1}{z}\sum_{m=0}^{M}\left(\frac{\zeta}{z}\right)^m d\zeta \\
&+ \frac{1}{2\pi i}\oint_{C_{R_1}} f(\zeta)\frac{1}{z}\frac{\left(\dfrac{\zeta}{z}\right)^{M+1}}{1-\dfrac{\zeta}{z}}\, d\zeta \\
&= \sum_{m=0}^{\infty}\frac{1}{z^{m+1}}\left(\frac{1}{2\pi i}\oint_{C_{R_2}} f(\zeta)\zeta^m\, d\zeta\right).
\end{aligned}$$

Note that $f_2(z)$ is analytic in the region $\{z\,||z| \ge R_1\}$.

The Laurent Expansion

Theorem. *Let $f(z)$ be analytic in the region $\{z\,|R_1 < |z| < R_2\}$. Then for $R_1 < |z| < R_2$,*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n,$$
$$a_n = \frac{1}{2\pi i}\oint_{C_r} f(\zeta)\zeta^{-n-1}\,d\zeta.$$

Here r is any number such that $R_1 < r < R_2$.
The series

$$f_1(z) = \sum_{n=0}^{\infty} a_n z^n$$

is analytic in $\{z\,||z| < R_2\}$,
The series

$$f_2(z) = \sum_{n=-\infty}^{-1} a_n z^n$$

is analytic in $\{z\,|R_1 < |z|\}$.

Consequences and Notes

- If $f(z)$ be analytic in the region $\{z\,||z| < R_2\}$, then

$$a_n = \frac{1}{2\pi i}\oint_{C_r} f(\zeta)\zeta^{-n-1}\,d\zeta = 0, n = -1, -2, \ldots.$$

- If $f(z)$ be analytic in the region $\{z\,|0 < |z| < R_2\}$, then

$$a_{-1} = \frac{1}{2\pi i}\oint_{C_r} f(\zeta)\,d\zeta$$

is called the *residue* of $f(z)$ at $z = 0$.

Zeroes, Poles, and Essential Singularities

For the moment, we shall consider a function $f(z)$ analytic in the *punctured disk*

$$\dot{D}_R = \{z\,|0 < |z| \le R\}.$$

Then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n,$$
$$a_n = \frac{1}{2\pi i}\oint_{C_r} f(\zeta)\zeta^{-n-1}\,d\zeta.$$

- If $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $f(z)$ may be extended by defining $f(0) = a_0$, and the resulting function is analytic in $|z| \le R$.

- If $f(z) = \sum_{n=N}^{\infty} a_n z^n$, $N \geq 0$, $a_N \neq 0$, $f(z)$ is said to have a *zero of order* N at $z = 0$. Near $z = 0$,
$$f(z) = z^N \cdot g(z)$$
, where $g(z)$ is analytic in $|z| \leq R$, $g(0) \neq 0$.
- If $f(z) = \sum_{n=-M}^{\infty} a_n z^n$, $M \geq 0$, $a_{-M} \neq 0$, $f(z)$ is said to have a *pole of order* M at $z = 0$. Near $z = 0$,
$$f(z) = z^{-M} \cdot g(z)$$
, where $g(z)$ is analytic in $|z| \leq R$, $g(0) \neq 0$.
- If $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, $a_n \neq 0$ for infinitely many negative n, then $f(z)$ is said to have an *essential singularity* at $z = 0$.
- The coefficient of z^{-1} is called the *residue* of $f(z)$ at $z = 0$, and is written
$$\text{Res}(f, z=0) = \text{Res} f(z)\big|_{z=0} = \frac{1}{2\pi i} \oint_{C_r} f(\zeta)\, d\zeta.$$

Exercises

1. Let $f(z)$ be analytic in the *punctured disk*
$$\dot{D}_R = \{z \,| 0 < |z| \leq R\}.$$
Then for r small and positive,
$$\oint_{C_r} f(\zeta)\, d\zeta = 2\pi i \, \text{Res} f(z)\big|_{z=0}.$$
2. Let $f(z)$ be analytic in the *punctured disk*
$$\dot{D}_R = \{z \,| 0 < |z| \leq R\}.$$
Suppose that $f(z)$ has a zero of order $N > 0$, at $z = 0$.
Then for r small and positive,
$$\oint_{C_r} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta = 2\pi i \cdot N.$$
3. Let $f(z)$ be analytic in the *punctured disk*
$$\dot{D}_R = \{z \,| 0 < |z| \leq R\}.$$
Suppose that $f(z)$ has a pole of order $M > 0$, at $z = 0$.
Then for r small and positive,
$$\oint_{C_r} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta = -2\pi i \cdot M.$$
4. Let $f(z)$ be analytic in the *punctured disk*
$$\dot{D}_R = \{z \,| 0 < |z| \leq R\}.$$
Suppose that $f(z)$ is bounded as $z \to 0$. Show that
- $\lim_{z \to 0} f(z)$ exists.
- $f(z)$ may be extended to be an analytic function in
$$D_R = \{z \,||z| \leq R\}.$$
As a consequence, the *singularity* of $f(z)$ at $z = 0$ is *removable*.

5. Let $f(z)$ be analytic in the *punctured disk*

$$\dot{D}_R = \{z \,|0 < |z| \leq R\}.$$

Suppose that

$$f(z) = O\left(|z|^M\right) \text{ as } z \to 0.$$

Show that for $n < M$, $a_n = 0$.