Residues at Isolated Singularities

If z_0 is a complex number we shall use the notation $C_r(z_0)$ for the closed circular path

$$C_r(z_0) = \{z \,||z - z_0| = r\}$$

traversed in the counterclockwise direction.

Residue Theorem – One Singularity Version. *Let $f(z)$ be analytic in the region $\{z\,|0 < |z - z_0| < R\}$. Then for z inside $C_r(z_0)$, $z \neq z_0$,*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n \,(z - z_0)^n \,,$$
$$a_n = \frac{1}{2\pi i} \oint_{C_r(z_0)} f(\zeta)\,(\zeta - z_0)^{-n-1} \, d\zeta.$$

Here r is any number such that $0 < r < R$.

In particular

$$a_{-1} = \frac{1}{2\pi i} \oint_{C_r(z_0)} f(\zeta)\, d\zeta,$$

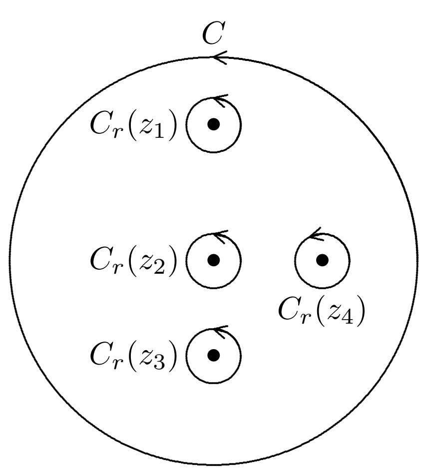
is the residue of $f(z)$ at $z = z_0$. so that

$$\oint_{C_r(z_0)} f(\zeta)\, d\zeta = 2\pi i \,\mathrm{Res}\, f(z)\big|_{z=z_0} .$$

The Residue Theorem

Let C be a simple closed path. Suppose that $f(z)$ is analytic on and inside C, except for a finite number of isolated singularities, $z_1, z_2, \ldots, z_K$ inside C.

Then, by using cuts from C to small circles of small radius r around each z_k,



$$\begin{aligned}\oint_C f(\zeta)\,d\zeta &= \sum_{k=1}^{K} \oint_{C_r(z_k)} f(\zeta)d\zeta \\ &= \sum_{k=1}^{K} 2\pi i \,\mathrm{Res}\; f(z)\big|_{z=z_k} \\ &= 2\pi i \sum_{k=1}^{K} \mathrm{Res}\; f(z)\big|_{z=z_k}\,.\end{aligned}$$

The Residue Theorem. *Let C be a simple closed path. Suppose that $f(z)$ is analytic on and inside C, except for a finite number of isolated singularities, $z_1, z_2, \ldots, z_K$ inside C. Then*

$$\oint_C f(\zeta)\,d\zeta = 2\pi i \sum_{k=1}^{K} \mathrm{Res}\; f(z)\big|_{z=z_k}\,.$$

The Residue Theorem reduces the problem of evaluating a *contour integral* – an integral on a simple closed path – to the algebraic problem of determining the poles and residues[1] of a function.

Exercises

Note the following special cases:

1. Let C be a simple closed path. Suppose that $f(z)$ is analytic on and inside C. Use the Residue Theorem to show that
$$\oint_C f(\zeta)\,d\zeta = 0.$$
We knew this result already!
2. Let C be a simple closed path. Let z be a point *inside* C. Find
$$\oint_C \frac{1}{\zeta - z}\,d\zeta.$$
3. Let C be a simple closed path. Let z be a point *outside* C. Find
$$\oint_C \frac{1}{\zeta - z}\,d\zeta.$$
4. Let C be a simple closed path. Let a and b be points *inside* C, $a \neq b$. Find
$$\oint_C \frac{1}{(\zeta - a)\,(\zeta - b)}\,d\zeta.$$

Finding the Residue

[1] Since functions behave so badly at an essential singularity, there is little hope of finding the residue at an essential singularity.

If $f(z)$ has a pole of order M at $z = z_0$, then

$$f(z) = \frac{a_{-M}}{(z - z_0)^M} + \ldots + \frac{a_{-1}}{z - z_0} + \ldots,$$
$$(z - z_0)^M f(z) = a_{-M} + a_{-M+1}(z - z_0) + \ldots + a_{-1}(z - z_0)^{M-1} + \ldots.$$

Since $(z - z_0)^M f(z)$ is analytic at $z = z_0$,

$$a_{-1} = \frac{1}{(M-1)!} \frac{d^{M-1}}{dz^{M-1}} (z - z_0)^M f(z) \Big|_{z=z_0}.$$

Partial Fractions

To find the partial fraction expansion of

$$\begin{aligned} f(z) &= \frac{q_{M+N-1}(z)}{(z - z_0)^M p_N(z)}, \\ \text{degree } q_{M+N-1} &= M + N - 1, \\ \text{degree } p_N &= N, \\ q_{M+N-1}(z_0) &\neq 0, \\ p_N(z_0) &\neq 0, \end{aligned}$$

let

$$h(z) = \frac{q_{M+N-1}(z)}{p_N(z)}.$$

Near $z = z_0$,

$$\begin{aligned} f(z) = &\frac{h(z_0)}{0!} \frac{1}{(z - z_0)^M} + \frac{h'(z_0)}{1!} \frac{1}{(z - z_0)^{M-1}} \\ &+ \ldots + \frac{h^{(M-1)}(z_0)}{(M-1)!} \frac{1}{z - z_0} + \text{analytic}. \end{aligned}$$

Similar expansions may be found near the roots of $p_N(z) = 0$.