

## Residues at Isolated Singularities

If  $z_0$  is a complex number we shall use the notation  $C_r(z_0)$  for the closed circular path

$$C_r(z_0) = \{z \mid |z - z_0| = r\}$$

traversed in the counterclockwise direction.

**Residue Theorem – One Singularity Version.** Let  $f(z)$  be analytic in the region  $\{z \mid 0 < |z - z_0| < R\}$ . Then for  $z$  inside  $C_r(z_0)$ ,  $z \neq z_0$ ,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$
$$a_n = \frac{1}{2\pi i} \oint_{C_r(z_0)} f(\zeta) (\zeta - z_0)^{-n-1} d\zeta.$$

Here  $r$  is any number such that  $0 < r < R$ .

In particular

$$a_{-1} = \frac{1}{2\pi i} \oint_{C_r(z_0)} f(\zeta) d\zeta,$$

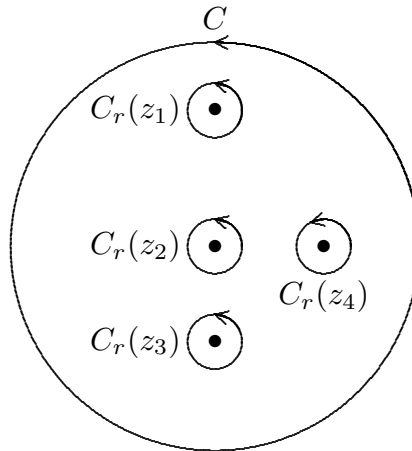
is the residue of  $f(z)$  at  $z = z_0$ . so that

$$\oint_{C_r(z_0)} f(\zeta) d\zeta = 2\pi i \operatorname{Res} f(z)|_{z=z_0}.$$

## The Residue Theorem

Let  $C$  be a simple closed path. Suppose that  $f(z)$  is analytic on and inside  $C$ , except for a finite number of isolated singularities,  $z_1, z_2, \dots, z_K$  inside  $C$ .

Then, by using cuts from  $C$  to small circles of small radius  $r$  around each  $z_k$ ,



$$\begin{aligned}
\oint_C f(\zeta) d\zeta &= \sum_{k=1}^K \oint_{C_r(z_k)} f(\zeta) d\zeta \\
&= \sum_{k=1}^K 2\pi i \operatorname{Res} f(z)|_{z=z_k} \\
&= 2\pi i \sum_{k=1}^K \operatorname{Res} f(z)|_{z=z_k}.
\end{aligned}$$

**The Residue Theorem.** Let  $C$  be a simple closed path. Suppose that  $f(z)$  is analytic on and inside  $C$ , except for a finite number of isolated singularities,  $z_1, z_2, \dots, z_K$  inside  $C$ . Then

$$\oint_C f(\zeta) d\zeta = 2\pi i \sum_{k=1}^K \operatorname{Res} f(z)|_{z=z_k}.$$

The Residue Theorem reduces the problem of evaluating a *contour integral* – an integral on a simple closed path – to the algebraic problem of determining the poles and residues<sup>1</sup> of a function.

### Exercises

Note the following special cases:

1. Let  $C$  be a simple closed path. Suppose that  $f(z)$  is analytic on and inside  $C$ . Use the Residue Theorem to show that

$$\oint_C f(\zeta) d\zeta = 0.$$

We knew this result already!

2. Let  $C$  be a simple closed path. Let  $z$  be a point *inside*  $C$ . Find

$$\oint_C \frac{1}{\zeta - z} d\zeta.$$

3. Let  $C$  be a simple closed path. Let  $z$  be a point *outside*  $C$ . Find

$$\oint_C \frac{1}{\zeta - z} d\zeta.$$

4. Let  $C$  be a simple closed path. Let  $a$  and  $b$  be points *inside*  $C$ ,  $a \neq b$ . Find

$$\oint_C \frac{1}{(\zeta - a)(\zeta - b)} d\zeta.$$

### Finding the Residue

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<sup>1</sup> Since functions behave so badly at an essential singularity, there is little hope of finding the residue at an essential singularity.

If  $f(z)$  has a pole of order  $M$  at  $z = z_0$ , then

$$f(z) = \frac{a_{-M}}{(z - z_0)^M} + \dots + \frac{a_{-1}}{z - z_0} + \dots,$$

$$(z - z_0)^M f(z) = a_{-M} + a_{-M+1}(z - z_0) + \dots + a_{-1}(z - z_0)^{M-1} + \dots$$

Since  $(z - z_0)^M f(z)$  is analytic at  $z = z_0$ ,

$$a_{-1} = \frac{1}{(M - 1)!} \frac{d^{M-1}}{dz^{M-1}} (z - z_0)^M f(z) \Big|_{z=z_0}.$$

## Partial Fractions

To find the partial fraction expansion of

$$f(z) = \frac{q_{M+N-1}(z)}{(z - z_0)^M p_N(z)},$$

degree  $q_{M+N-1} = M + N - 1$ ,  
degree  $p_N = N$ ,  
 $q_{M+N-1}(z_0) \neq 0$ ,  
 $p_N(z_0) \neq 0$ ,

let

$$h(z) = \frac{q_{M+N-1}(z)}{p_N(z)}.$$

Near  $z = z_0$ ,

$$f(z) = \frac{h(z_0)}{0!} \frac{1}{(z - z_0)^M} + \frac{h'(z_0)}{1!} \frac{1}{(z - z_0)^{M-1}}$$

$$+ \dots + \frac{h^{(M-1)}(z_0)}{(M - 1)!} \frac{1}{z - z_0} + \text{analytic}.$$

Similar expansions may be found near the roots of  $p_N(z) = 0$ .