The Laurent Expansion

Theorem. *Let $f(z)$ be analytic in the region $\{z\,|0<|z|<R\}$. Then for $0<|z|<R$,*

$$f(z)=\sum_{n=-\infty}^{\infty} a_n z^n,$$

$$a_n=\frac{1}{2\pi i}\oint_{C_r} f(\zeta)\zeta^{-n-1}\,d\zeta.$$

Here r is any number such that $0<r<R$.

The series

$$f_1(z)=\sum_{n=0}^{\infty} a_n z^n$$

is analytic in $\{z\,||z|<R\}$,

The series

$$f_1(z)=\sum_{n=-\infty}^{-1} a_n z^n$$

is analytic in $\{z\,|0<|z|\}$.

Note that

- If $f(z)$ be analytic in the region $\{z\,||z|<R\}$, then

$$a_n=\frac{1}{2\pi i}\oint_{C_r} f(\zeta)\zeta^{-n-1}\,d\zeta=0, n=-1,-2,\ldots.$$

- If $f(z)$ be analytic in the region $\{z\,|0<|z|<R\}$, then

$$a_{-1}=\frac{1}{2\pi i}\oint_{C_r} f(\zeta)\,d\zeta$$

is called the *residue* of $f(z)$ at $z=0$.

Isolated Singularities

For the moment, we shall consider a function $f(z)$ analytic in the *punctured disk*

$$\dot{D}_R=\{z\,|0<|z|\le R\}.$$

Thus the possible singularity of $f(z)$ at $z=0$ is *isolated*.

Then

$$f(z)=\sum_{n=-\infty}^{\infty} a_n z^n,$$

$$a_n=\frac{1}{2\pi i}\oint_{C_r} f(\zeta)\zeta^{-n-1}\,d\zeta.$$

- The coefficient of z^{-1} is called the *residue* of $f(z)$ at $z=0$, and is written

$$\text{Res}(f,z=0)=\text{Res}f(z)|_{z=0}=\frac{1}{2\pi i}\oint_{C_r} f(\zeta)\,d\zeta.$$

- If $f(z)=\sum_{n=0}^{\infty} a_n z^n$, $f(z)$ may be extended by defining $f(0)=a_0$, and the resulting function is analytic in $|z|\le R$. In this case the singularity is *removable*.

- If $f(z) = \sum_{n=N}^{\infty} a_n z^n$, $N \geq 0$, $a_M \neq 0$, $f(z)$ is said to have a *zero of order* N at $z = 0$. Near $z = 0$,
$$f(z) = z^N \cdot g(z)$$
, where $g(z)$ is analytic in $|z| \leq R$, $g(0) \neq 0$.
- If $f(z) = \sum_{n=-M}^{\infty} a_n z^n$, $M \geq 0$, $a_{-M} \neq 0$, $f(z)$ is said to have a *pole of order* M at $z = 0$. Near $z = 0$,
$$f(z) = z^{-M} \cdot g(z)$$
, where $g(z)$ is analytic in $|z| \leq R$, $g(0) \neq 0$. The function is also *meromorphic* in $\dot{D}_R$.
- If $f(z) = O\left(|z|^{-M}\right)$, $M \geq 0$, the preceding exercises show that $a_{-M-1} = 0$, $a_{-M-2} = 0$, $\ldots$. Thus $f(z)$ at $z = 0$ has a pole of order *at most* M.
- At $z = 0$, $f(z)$ has a pole of order M iff there are positive constants c_1 and c_2 such that
$$\frac{c_1}{|z|^M} \leq |f(z)| \leq \frac{c_2}{|z|^M}.$$

Isolated Essential Singularities

Definition. *If $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, $a_n \neq 0$ for infinitely many negative n, then $f(z)$ is said to have an essential singularity at $z = 0$.*

Analytic functions which have isolated essential singularities behave very badly near the essential singularity.

Theorem (Little Picard). *Suppose that $f(z)$ has an essential singularity at $z = 0$. Then for any complex number w_0, in any neighborhood of $z = 0$, $f(z)$ gets arbitrarily close to w_0.*

Proof of the Little Picard Theorem: The proof is by contradiction.If there is a neighborhood $\dot{D}_r = \{z\,|0 < |z| < r\}$ in which $f(z) - w_0$ is bounded away from 0, then
$$g(z) = \frac{1}{f(z) - w_0}$$
is analytic and bounded in $\dot{D}_r$. Thus $g(z)$ has a removable singularity at $z = 0$ and a zero of order N, $N \geq 0$. Thus $g(z) = z^N \cdot h(z)$, $h(z)$ analytic near $z = 0$ and $h(0) \neq 0$. Possibly shrinking r, we may assume that $h(z) \neq 0$ in $D_r = \{z\,||z| < r\}$. Then
$$f(z) - w_0 = z^{-N} \cdot \frac{1}{h(z)}.$$
It follows that $f(z)$ has a pole of order at most N at $z = 0$.