Sums and Unconditional Convergence

For manipulations with power series we are interested in infinite sums such as

$$\sum_{j,k=0}^{\infty} a_{j,k}.$$

We need to justify *changing the order of summation* and conditions which assure that

$$\sum_{j,k=0}^{\infty} a_{j,k} = \sum_{j=0}^{\infty}\left(\sum_{k=0}^{\infty} a_{j,k}\right) = \sum_{j=0}^{\infty}\left(\sum_{k=0}^{\infty} a_{j,k}\right).$$

The condition which justifies the manipulations is *absolute* or *unconditional convergence* of the series.

Theorem. *Suppose that all the finite [partial] sums*

$$\sum_{j,k\in \text{finite set}} |a_{j,k}|$$

form a bounded set. Then there is a unique finite (complex) number

$$S = \sum_{j,k=0}^{\infty} a_{j,k}$$

which may be calculated as

$$\sum_{j,k=0}^{\infty} a_{j,k} = \sum_{j=0}^{\infty}\left(\sum_{k=0}^{\infty} a_{j,k}\right) = \sum_{j=0}^{\infty}\left(\sum_{k=0}^{\infty} a_{j,k}\right),$$

or with any rearrangement in the summation of the terms – for example

$$S = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} a_{k,k-j}\right).$$

Since the sum may be calculated in "any order", the convergence is called *unconditional*.

Notes on Unconditional Convergence

Let A be an index set[1]. For each $\alpha \in A$, Let z_α be a complex number. Then the sum

$$\sum_{\alpha \in A} z_\alpha$$

converges unconditionally to S if *almost all the finite sums are close to* S in the precise sense:

Given $\epsilon > 0$, there is a finite set F_ϵ such that if F is any finite subset of A, $F \supseteq F_\epsilon$, then

$$\left| \sum_{\alpha \in F} z_\alpha - S \right| < \epsilon.$$

Note that if $\sum_{\alpha \in A} z_\alpha$ converges unconditionally to S, then $\sum_{\alpha \in A} \Re z_\alpha$ converges unconditionally to $\Re S$ and $\sum_{\alpha \in A} \Im z_\alpha$ converges unconditionally to $\Im S$.

In particular, if $z_\alpha = x_\alpha \geq 0$, then

$$S = \sup_{F \text{ finite}} \sum_{\alpha \in F} x_\alpha .$$

In this case $\sum_{\alpha \in A} x_\alpha$ converges unconditionally iff the set of finite sums $\sum_{\alpha \in F} x_\alpha$, F finite, is bounded.

Let x_α be real. Define

$$\begin{aligned} x_\alpha^+ &= \max\left(x_\alpha, 0\right), \\ x_\alpha^- &= \max\left(-x_\alpha, 0\right), \end{aligned}$$

so that

$$\begin{aligned} x_\alpha &= x_\alpha^+ - x_\alpha^-, \\ |x_\alpha| &= x_\alpha^+ + x_\alpha^-. \end{aligned}$$

[1] For technical reasons, if $A = \emptyset$, the empty set, interpret $\sum_{\alpha \in \emptyset} z_\alpha = 0$.

Theorem. *Let x_α be real. The sum* $\sum_{\alpha\in A} x_\alpha$ *converges unconditionally iff the sums* $\sum_{\alpha\in A} x_\alpha^+$, $\sum_{\alpha\in A} x_\alpha^-$, *and* $\sum_{\alpha\in A} |x_\alpha|$ *all converge unconditionally.*

Proof. There is a finite set F_1 such that $\left|\sum_{\alpha\in F, F\supseteq F_1} x_\alpha - S\right| < 1$. Then if F is any finite set,

$$\begin{aligned}\sum_{\alpha\in F} x_\alpha^+ &\le \sum_{\alpha\in F\setminus F_1} x_\alpha^+ + \sum_{\alpha\in F\cap F_1} x_\alpha^+\\ &\le 1 + \sum_{\alpha\in F_1} |x_\alpha|\\ &\le C.\end{aligned}$$

Thus all the finite sums of $\sum_{\alpha\in A} x_\alpha^+$, $\sum_{\alpha\in A} x_\alpha^-$, and $\sum_{\alpha\in A} |x_\alpha|$ are bounded.

Summary

By considering $z_\alpha = x_\alpha^+ - x_\alpha^- + i\,(y_\alpha^+ - y_\alpha^-)$, we obtain:

1. The sum $\sum_{\alpha\in A} z_\alpha$ converges unconditionally iff

- The sum $\sum_{\alpha\in A} |z_\alpha|$ converges unconditionally.

  iff

- The set of finite sums $\sum_{\alpha\in F} |z_\alpha|$, F finite, is bounded.

2. If $\{A_j\}$ is a sequence of sets increasing to A ($A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots, A = \cup_{j=0}^\infty A_j$), it is easy to see that if $\sum_{\alpha\in A} z_\alpha$ converges unconditionally, then

$$\sum_{\alpha\in A} z_\alpha = \lim_{j\to\infty} \sum_{\alpha\in A_j} z_\alpha.$$

3. **Rearrangement:** If $\sum_{\alpha\in A} z_\alpha$ converges unconditionally, and A is a disjoint union of sets,

$A = \cup_{j=0}^{\infty} B_j$, with the B_j pairwise disjoint, then

$$\sum_{\alpha \in A} z_\alpha = \sum_{j=0}^{\infty} \left(\sum_{\alpha \in B_j} z_\alpha \right).$$