MthT 430 Notes Chapter 11b Consequences of MVT

There are several important consequences of Rolle's Theorem and the Mean Value Theorem (MVT).

Mean Value Theorem (MVT). *If f is continuous on (a, b) and differentiable on $[a, b]$, then there is an x in (a, b) such that*

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

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Corollary 1, p. 191. *If f is defined on an interval I, and $f'(x) = 0$ on I, then f is constant on I.*

This corollary is important in integration – Two functions F, G which have the same derivative on an interval must differ by a constant C on the interval.

Proof. For any a, b, in I,

$$f(b) - f(a) = f'(c) \cdot (b - a) = 0.$$

Theorem. *If f is continuous and differentiable on an interval I, and $f' > 0$ on I, then f is* **increasing** *on I.*

Proof. For a, b in I,

$$f(b) - f(a) = f'(c) \cdot (b - a)\,,$$

so that $f(b) - f(a)$ has the same sign as $b - a$. Note that the conclusion holds if we assume that f is continuous on I and that for x not an endpoint of I, $f'(x) > 0$.

The following is known as the *First Derivative Test for a Local Maximum/Minimum.*

Theorem. *If a number a in an interval I is a critical point of a function f, and there is a $\delta > 0$, such that for $0 < x - a < \delta$, $f'(x) > 0$, and for $-\delta < x - a < 0$, $f'(x) < 0$, then a is a local minimum point for f on I.*

Proof. For $0 < |x - a| < \delta$,

$$\begin{aligned} f(x) - f(a) &= f'(c_x) \cdot (x - a) \\ &> 0, \end{aligned}$$

since c_x is between a and x.

The Mean Value Theorem may also be used to prove that derivatives exist.

Theorem. *Suppose that* f *is continuous at* a *and that* $\lim_{x\to a} f'(x)$ *exists. Then* $f'(a)$ *exists and*

$$f'(a) = \lim_{x\to a} f'(x).$$

Proof.

$$\begin{aligned} \lim_{x\to a} \frac{f(x)-f(a)}{x-a} &= \lim_{x\to a} f'(c_x) \\ &= \lim_{x\to a} f'(x). \end{aligned}$$

N.B. – Be Careful! The Theorem does not say that all functions which are derivatives are also continuous.