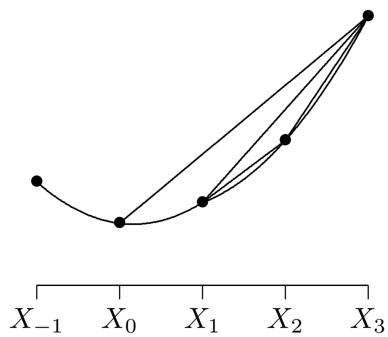
MthT 430 Notes Derivatives of Convex Functions

A function f is *convex* on an interval I if every secant line is above the graph on I.



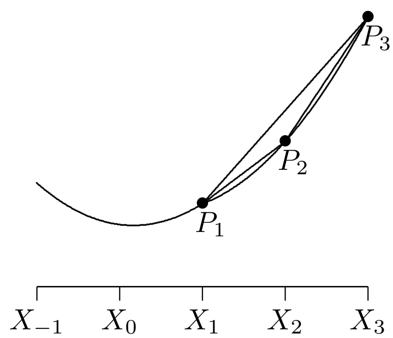
Algebraically, convexity is expressed by: If $X_1 < X_2$, then for $X_1 < X < X_2$,

$$f(X) \leq f(X_1) + \frac{f(X_2) - f(X_1)}{X_2 - X_1}\,(X - X_1)\,.$$

1. Show that if $X_1 < X_2 < X_3$, then

$$\frac{f(X_2) - f(X_1)}{X_2 - X_1} \leq \frac{f(X_3) - f(X_1)}{X_3 - X_1},$$
$$\frac{f(X_3) - f(X_1)}{X_3 - X_1} \leq \frac{f(X_3) - f(X_2)}{X_3 - X_2}.$$

This is an algebraic verification of the geometric observation that slope$\overline{P_1 P_2} \leq$ slope$\overline{P_1 P_3}$ and slope$\overline{P_1 P_3} \leq$ slope$\overline{P_2 P_3}$.



In particular, as X_3 *decreases* to X_2, the difference quotient

$$\frac{f(X_3) - f(X_2)}{X_3 - X_2}$$

decreases and is bounded below by

$$\frac{f(X_2) - f(X_1)}{X_2 - X_1}.$$

Thus

$$\lim_{X \to X_2^+} \frac{f(X) - f(X_2)}{X - X_2} = D_+ f(X_2)$$

exists. The right hand limit of the difference quotient,

$$D_+ f(x) = \lim_{\Delta x \to 0^+} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

is called the *right derivative* or *right derivate* of f at x.

Similarly the left derivative

$$\lim_{X \to X_2^-} \frac{f(X) - f(X_2)}{X - X_2} = D_- f(X_2)$$

exists (and is $\leq D_+ f(X_2)$).

Thus we have shown: If f is convex on an open interval $I = (a, b)$, then for each $x \in I$, the left derivative $D_- f(x)$ and the right derivative $D_+ f(x)$ both exist. If they are the same, then $f'(x_0)$ exists. Note that if $a < x_1 < x_2 < b$, then

$$D_- f(x_1) \leq D_+ f(x_1) \leq D_- f(x_2) \leq D_+ f(x_2).$$

It follows that $D_- f(x)$ and $D_- f(x)$ are *nondecreasing* functions on I.

2. Is the following result true? If not, give a counterexample.

Theorem. *If f is convex on an open interval $I = (a, b)$, then f is continuous on I.*