MthT 430 Notes Chapter 4b Binary Expansions and Graphs

Construction of the Binary Expansion by Recursion

Let x, $0 \le x < 1$, be a real number.

We construct the binary expansion of x:

$$x = 0._{\text{bin}} b_1 b_2 \ldots,$$
$$b_k \in \{0, 1\}.$$

This is the statement the infinite series

$$b_1 2^{-1} + b_2 2^{-2} + \ldots + b_k 2^{-k} + \ldots, \quad b_k \in \{0, 1\},$$

converges to x.

Step 1. Divide the interval $\left[0, \frac{1}{2^0}\right)$ into the two halves $\left[0, \frac{1}{2^1}\right)$ and $\left[\frac{1}{2^1}, \frac{1}{2^0}\right)$.

If $0 \le x < \frac{1}{2^1}$, let

$$\begin{cases} b_1 = 0, \\ s_1 = 0._{\text{bin}} b_1, \\ r_1 = x - s_1. \end{cases}$$

If $\frac{1}{2^1} \le x < \frac{1}{2^0}$, let

$$\begin{cases} b_1 = 1, \\ s_1 = 0._{\text{bin}} b_1, \\ r_1 = x - s_1. \end{cases}$$

Then $0 \le r_1 < \frac{1}{2^1}$.

If $r_1 = 0$, for $n > 1$, define $b_n = 0$ and **Stop!** $x = s_1 = 0._{\text{bin}} b_1$.

Suppose that **Step 1.** ... **Step k.** have been completed so that $b_j = 0$ or 1 have been defined so that

$$\begin{cases} s_k = 0._{\text{bin}} b_1 \ldots b_k, \\ r_k = x - s_k, \\ 0 \le r_k < \frac{1}{2^k}. \end{cases}$$

Step (k+1). Divide the interval $\left[0, \frac{1}{2^k}\right)$ into the two halves $\left[0, \frac{1}{2^{k+1}}\right)$ and $\left[\frac{1}{2^{k+1}}, \frac{1}{2^k}\right)$.

If $0 \leq r_k < \dfrac{1}{2^{k+1}}$, let

$$\begin{cases} b_{k+1} = 0, \\ s_{k+1} = 0._{\text{bin}} b_1 \dots b_{k+1}, \\ r_{k+1} = x - s_{k+1}. \end{cases}$$

If $\dfrac{1}{2^{k+1}} \leq r_k < \dfrac{1}{2^k}$, let

$$\begin{cases} b_{k+1} = 1, \\ s_{k+1} = 0._{\text{bin}} b_1 \dots b_{k+1}, \\ r_{k+1} = x - s_{k+1}. \end{cases}$$

Then $0 \leq r_{k+1} < \dfrac{1}{2^{k+1}}$.

If $r_{k+1} = 0$, for $n > k+1$, define $b_n = 0$ and **Stop!** $x = s_{k+1} = 0._{\text{bin}} b_1 \dots b_{k+1}$.

Remark. Note that

$$\lim_{k\to\infty} s_k = \lim_{k\to\infty} (x - r_k) = x.$$