MthT 430 Projects Chap 8f – $\limsup$ and $\liminf$

- (See also Spivak Chapter 8 - Problem 18) Let $\{x_k\}$ be a bounded sequence. We define the *limit superior*($\limsup$) and *limit inferior*($\liminf$) of the sequence to be

$$\limsup_{k\to\infty} x_k = \lim_{k\to\infty}\left(\sup_{n\geq k} x_n\right),$$
$$\liminf_{k\to\infty} x_k = \lim_{k\to\infty}\left(\inf_{n\geq k} x_n\right).$$

- (P13–BISHL) shows that both $\limsup\limits_{k\to\infty} x_k$ and $\liminf\limits_{k\to\infty} x_k$ exist.

- Show that

$$\limsup_{k\to\infty} x_k = A$$

if and only if for every $\epsilon > 0$,

$$\begin{cases} x_k > A+\epsilon & \text{for at most finitely many } k, \\ x_k > A-\epsilon & \text{for infinitely many } k. \end{cases}$$

- Show that if $\{x_k\}$ and $\{y_k\}$ are bounded sequences, then

$$\limsup_{k\to\infty}(x_k+y_k) \leq \limsup_{k\to\infty} x_k + \limsup_{k\to\infty} y_k.$$

- Give an example of bounded sequences $\{x_k\}$ and $\{y_k\}$ such that

$$\limsup_{k\to\infty}(x_k+y_k) < \limsup_{k\to\infty} x_k + \limsup_{k\to\infty} y_k.$$

- Show that if $\{x_k\}$ and $\{y_k\}$ are bounded sequences, then

$$\liminf_{k\to\infty} x_k + \limsup_{k\to\infty} y_k \leq \limsup_{k\to\infty}(x_k+y_k).$$

- The general result is that for two bounded sequences $\{x_k\}$ and $\{y_k\}$,

$$\begin{aligned}\liminf_{k\to\infty} x_k + \liminf_{k\to\infty} y_k &\leq \liminf_{k\to\infty}(x_k+y_k) \\ &\leq \liminf_{k\to\infty} x_k + \limsup_{k\to\infty} y_k \\ &\leq \limsup_{k\to\infty}(x_k+y_k) \\ &\leq \limsup_{k\to\infty} x_k + \limsup_{k\to\infty} y_k.\end{aligned}$$