MthT 491 Mathematical Statements

Formulation of Mathematical Propositions

The following observations are motivated by the discussion in Chapter 1 of **An Introduction to the Theory of Numbers** by Ivan Niven and Herbert Zuckerman [**N–Z**].

Students (including me!) are often tripped by mathematical statements which are stated differently as a matter of convenience or style. To quote [**N–Z**]:

> ...if A denotes some assertion or collection of assertions, and B likewise, the following statements are equivalent – they are just different ways of saying the same thing.

- A implies B.
- If A is true, then B is true.
- In order that A be true it is necessary that B be true.
- B is a necessary condition for A.
- A is a sufficient condition for B.

We add the equivalent statements:

- If A, then B.
- Whenever A is true, B is true.
- Whenever A, B.
- A implies B.
- B is implied by A.
- $A \Longrightarrow B$.
- Satisfying A implies satisfying B.

Definitions and Variations

In calculus, an important definition is limit of a function $f(x)$, as x approaches ($\to$) a. I take the definition from Michael Spivak's **Calculus** [**S**].

Definition. *(Spivak, p. 96)*

$$\lim_{x \to a} f(x) = L$$

means: For every $\epsilon > 0$, *there is some* $\delta > 0$ *such that, for all* x, *if* $0 < |x - a| < \delta$, *then* $|f(x) - L| < \epsilon$.

Remarks. After many years of looking at students' rephrasing a definition, we wish to decide which variations are "correct" and still give an equivalent definition.

What is the point of the exercise? Think of a *Definition* as an *If and Only If Theorem*. Thus you are able to use interchangeably the phrases

- $\lim_{x \to a} f(x) = L$.
- For every $\epsilon > 0$, there is some $\delta > 0$ such that, for all x, if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

Replacing the names of the variables, we could use interchangeably the phrases

- $\lim_{h \to 0} f(x + h) = L$.
- For every $\epsilon > 0$, there is some $\delta > 0$ such that, for all h, if $0 < |h| < \delta$, then $|f(x + h) - L| < \epsilon$.

The actual details are:

1. The *function* of the *variable* h having a limit as $h \to 0$ is $f(x + h)$, or in a programming languages, $h \to f(x + h)$. The number x is an inert parameter in the definition of the function.
2. Using the definition: For every $\epsilon > 0$, there is some $\delta > 0$ such that, for all h, if $0 < |h - 0| < \delta$, then $|f(x + h) - L| < \epsilon$.

With experience, if needed we could also change the names of variables introduced internally within the descriptive statement (in this example ϵ, δ). Thus (Be careful!) we could say:

For every $\epsilon_1 > 0$, there is some $\delta_1 > 0$ such that, for all h, if $0 < |h| < \delta_1$, then $|f(x + h) - L| < \epsilon_1$.

Remarks. After many years of looking at students' rephrasing a definition, we wish to

decide which variations are "correct" and still give an equivalent definition.

The phrase "**Definition X** is *equivalent* to **Definition Y**" means you can use interchangeably the phrases

- [Definition] Term[1] (What is being defined)
- [Definition] Description X (Details)
- [Definition] Description Y (Details)

Now if **Definition X** is *not equivalent* to **Definition Y**, then *at least one* of the following is false:

- Definition Description X $\Rightarrow$ Definition Description Y.
- Definition Description Y $\Rightarrow$ Definition Description X.

Interpreting each of the above as a *Theorem*, the way to show a *Theorem* is false is to construct a *counterexample*. A *counterexample* is an object [construct, ...], which satisfies the hypotheses of the Theorem, but does not satisfy the conclusion[s] of the Theorem.

So let's begin.

[1] I borrow the words *Definition Term* and *Definition Description* from the `html` tags `<DT>` and `<DD>`.

Contradiction

If A denotes some assertion or collection of assertions, we have a *contradiction* if both A and $\neg A$, the *negation of* A are true.

A theorem

$$A \Rightarrow B$$

is *proved by contradiction* if we show that

$$\neg B \Rightarrow \neg A.$$

Please note that usually the assertion A may contain within itself many definitions and properties not stated explicitly. For example, if A contains the statement

n is a natural number $\ldots$,

and we proved that

$\neg B$ implies $n < 0$.

we would have a *proof by contradiction* of $A \Rightarrow B$.