Sample problems for Exam 2 ______ (2 pages!) ______ Math 310

1. (a) Find the matrix A that represents the linear operator L with respect to the standard basis $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$, where

$$L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x - y - z \\ x - 2y + z \\ -x + 3y - 2z \end{pmatrix}$$

(b) Find the kernel of L.

2. The subspace V of $\mathbb{R}^3$ is spanned by

$$\mathbf{u}_1 = (2,0,1)^T,\ \mathbf{u}_2 = (1,2,3)^T,\ \mathbf{u}_3 = (5,2,5)^T.$$

Find a basis for its orthogonal complement $V^\perp$.

3. Let $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$ be the standard basis of $\mathbb{R}^3$ and

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},\ \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},\ \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

The linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$L(\mathbf{v}_1) = \mathbf{v}_1,\ L(\mathbf{v}_2) = 2\mathbf{v}_2,\ L(\mathbf{v}_3) = 3\mathbf{v}_3.$$

(a) Find the coordinates of $\mathbf{e}_1$ with respect to the basis $[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$. Express $L(\mathbf{e}_1)$ in terms of the standard basis.

(b) What is the first column of the matrix representing L with respect to the standard basis?

(c) What is the matrix representing L with respect to the basis $[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$?

4. Let $L : P_3 \to P_3$ be the linear operator defined by

$$L(p) = p' + p$$

(a) Find the matrix A representing L in the standard basis $[1, x, x^2]$.

(b) Find the matrix B representing L in the basis $[1, x+1, x^2+1]$.

(c) Are A and B similar matrices? Explain.

(d) Find a matrix T such that $B = T^{-1}AT$.

5. Given two vectors $\mathbf{u} = (1,2,3)^T$ and $\mathbf{v} = (3,-1,2)^T$,

(a) Find $||\mathbf{u}||$, $||\mathbf{v}||$ and $\langle \mathbf{u}, \mathbf{v} \rangle$.

(b) Find the vector projection of $\mathbf{u}$ onto $\mathbf{v}$.

(c) What is the angle between these vectors?

6. Consider $\mathbf{p} = (3,3,3)$ and the plane $x - y + 3z = 0$.

(a) Find a unit normal vector to the plane.

(b) Find the distance from the point $\mathbf{p}$ to the plane.

(c) Find the projection of $\mathbf{p}$ onto the plane.

7. Find the equation of the line $y = c_0 + c_1 x$ which is a least squares best fit to the points $(1,2), (2,1), (4,-3), (5,-4)$.

8. Find the equation of the best quadratic fit, $y = c_0 + c_1 x + c_2 x^2$, to the points $(1, 2), (2, 1), (4, -3), (5, -4)$.

9. Find an orthonormal basis of the subspace of $\mathbb{R}^3$ of vectors orthogonal to $\mathbf{v} = (4, 3, -3)^T$.

10. Consider the subspace $V = \text{Span}(1, \cos x, \cos 2x)$ of $C[-\pi, \pi]$ equipped with the inner product defined by

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} fg\ dx$$

and an orthonormal basis $[\frac{1}{\sqrt{2}}, \cos x, \cos 2x]$ of V.

(a) Compute

$$\int_{-\pi}^{\pi} (2 + 3\cos 2x)(\cos x + \cos 2x)\ dx$$

(b) Compute $||2 + \cos x + \cos 2x||$.

(c) What is the cosine of the angle between $f = 2+3\cos 2x$ and $g = \cos x+\cos 2x$

11. Let V be the subspace of $\mathbb{R}^3$ spanned by

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix},\ \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},\ \mathbf{v}_3 = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}.$$

(a) Find a basis for the orthogonal space $V^{\perp}$.

(b) What are the dimensions of V and $V^{\perp}$?

(c) Find a basis for V.

(d) Use the Gram-Schmidt method to give an orthonormal basis for V.

(e) Use your answer to (a) to extend the orthonormal basis for V found in (d) to the orthonormal basis of $\mathbb{R}^3$.

12. Let

$$A = \begin{pmatrix} 3 & -1 \\ 4 & 2 \\ 0 & 2 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 0 \\ 20 \\ 10 \end{pmatrix}.$$

(a) Use the Gram-Schmidt method to give an orthonormal basis for the column space $R(A)$.

(b) Find a QR-factorization of A.

(c) Using (b) solve the least squares problem for $A\mathbf{x} = \mathbf{b}$.