# Review of the first two parts of the course

In the first two parts of the course we learned about Maple's advanced number system and about its favorite objects: polynomials and rational expressions.

As the final exam happens in a computer lab, all worksheets and lecture notes will still be available to you via the course web site. However, as there is always a slim chance that the network or web server fails, it is strongly recommended to have a well organized folder with lecture notes.

Below is a first, preliminary list of fresh questions to review. Consider also the review questions for the midterms, the quizzes, midterms, and homework assignments.

## Review of Part One: First Steps with Maple

1. Explain the difference between `evalhf(1.0+10^(-10))` and `evalhf(1+10^(-10))`.

   The machine precision of a floating-point number system is defined as the smallest number you can add to one and obtain a result that is still larger than one. Give the Maple command(s) to get the magnitude of the machine precision.

2. Consider $\mathbb{Z}_{31} = \{ 0, 1, 2, \ldots, 31 \}$ and answer the following questions:

   (a) What is the multiplicative inverse of 7 in $\mathbb{Z}_{31}$?

   (b) Show that $p = 15x^5 + 4x^4 + 23x^3 + 26x^2 + 6x + 1$ is irreducible over $\mathbb{Z}_{31}$.

   (c) Declare $\alpha$ as a formal root of $p$. How many elements has $\mathbb{Z}_{31}(\alpha)$? Justify your answer.

   (d) Compute the value $\alpha^{21}$ as an element in $\mathbb{Z}_{31}(\alpha)$.

3. Give all Maple commands to write $e^{I \frac{2\pi}{k}}$ as $\cos(2\frac{\pi}{k}) + I \sin(2\frac{\pi}{k})$.

4. The sequence `restart; s := a+b: a := x+y: b := u+v: s;` shows `x+y+u+v`.

   (a) Give the Maple command to show that Maple still knows that $s = a + b$.

   (b) Give one single Maple command to change `s` so that typing `s` shows `x + y + u + v + c`.

5. Illustrate a good use of the **assign** command.

   Give an example of a Maple session in which the outcomes of **assign(x,5)** and **x := 5** are different.

6. Consider the expression $q = \cos(x^3 - 1) + 3\sin(y) - z^7$. Draw the expression tree for $q$ and give all Maple commands you used to make the drawing.

7. How do we bring a matrix of floating-point numbers from file into a Maple session?
   Illustrate with a good example.

8. Generate optimized code to evaluate $p = 79x^{298} + 56x^{205} + 49x^{164} + 63x^{121} + 57x^{119} - 59x^{42}$.
   How many aritmetical operations are needed to evaluate $p$?
   Compare with the cost of a direct evaluation of $p$.

### Review of Part Two: Polynomials and Rational Expressions

9. Draw the internal representation of $p := xy(x-y)$. Give also the Maple command(s) (but not the output!) used to obtain your drawing. Explain why **subs(1=-1,p)** returns $\frac{1}{xy(-x-y)}$.

10. Consider the polynomial $p = x^3 - x - 2$ and give all Maple commands following questions:
    (a) to write $p$ as an **exact** product of linear factors, with exact complex numbers;
    (b) to compute a **numerical** factorization of $p$ over the complex numbers;
    (c) to define a **symbolic** (i.e.: formal) factorization of $p$, declaring sufficiently many roots.

11. Give all Maple commands to transform $(x-y)(x+y)$ into $(x+y)x - (x+y)y$.

12. Consider the rational expression $r = \frac{79x^5+56x^4+49x^3+63x^2+57x-59}{45x^5-8x^4-93x^2+43x-62}$.
    Convert $r$ into a form which is more efficient to evaluate. Compare the number of arithmetical operations needed to evaluate $r$ in this more efficient form with the number of arithmetical operations needed to evaluate $r$ in its given form.

13. Explain why normal forms are so important to symbolic computation.
    What can we do if a normal form is too expensive to compute? Illustrate with a good example.

## Review of parts three and four of the course

In part three of the course we saw how to define functions and explored Maple's capabilities in differentiation and integration. We discussed Maple's data structures and solved equations in the fourth and final part on Maple.

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Below is a first, preliminary list of fresh questions to review. Consider also the review questions for the midterms, the quizzes, midterms, and homework assignments.

1. Use piecewise to define a function `int_inv_cub` which as function of the end points $a$ and $b$ always returns the correct value of $\int_a^b \frac{1}{x^3} dx$.

2. How can you make Maple return $5x^4 dx$ as a result of the command `diff(x^4,x)`?

3. The arc length of continuous function $f(x)$ over an interval $[a,b]$ can be defined as $\int_a^t \sqrt{1+[f'(x)]^2}$.
    (a) Compute the arc length of the positive half of the unit circle, i.e.: $f(x) = \sqrt{1-x^2}$ (answer $= \pi$).
    (b) Create a function (call it `arc_length`) in $t$ which returns a 10-digit floating-point approximation of the arc length of the positive half of the circle, for $x \in [0,t]$.

4. Consider the recurrence relation

$$h(n) = 5h(n-1) - 6h(n-2), \quad \text{for } n \geq 2, \quad \text{with } h(0) = 1 \text{ and } h(1) = -2.$$

    (a) The generating function $g(x) = \frac{1-7x}{1-5x+6x^2}$ defines $h(n)$ as the coefficient with $x^n$ in the Taylor expansion of $g(x)$. Use $g(x)$ to define $h$ as a function (call it `t`) of $n$ which gives the value of $h(n)$.

(b) Write a procedure to compute $h(n)$, directly using the recurrence relation from above. Make sure your procedure can compute $h(120)$. Compare with the result of (a).

(c) Find an explicit expression for $h(n)$ as a function of $n$. Use this expression to define a function `s` which returns $h(n)$. Compare `s`(120) with `t`(120) and $h(120)$.

5. The Legendre polynomials are defined by

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_n(x) = \frac{2n-1}{n} x P_{n-1}(x) - \frac{n-1}{n} P_{n-2}(x), \text{ for } n \geq 2.$$

Write a efficient recursive procedure `legendre` to compute $P_n(x)$. The procedure `legendre` takes on input the variable $x$ and as index the degree $n$ of $P_n$.
Compare the output of `legendre`[50](x) with `orthopoly[P](50,x)`.

6. What is the main difference between automatic and symbolic differentiation? In which circumstances do we prefer the result of an automatic differentiation? Give a good example to illustrate your point.

7. The command **stats[random,uniform[-1,1]]** allows us to generate random numbers, uniformly distributed in $[-1, 1]$.

(a) Create two lists, call them `lx` and `ly`, each with 100 hundred random points, uniformly sampled from $[-1, 1]$.

(b) Join the two lists `lx` and `ly` into one list of points. Each point is represented by a list of two coordinates, one from the list `lx`, and the other from the list `ly`.

(c) Select from the list of points all points inside the unit circle $x^2 + y^2 = 1$.
How many such points do you find?

8. Suppose we want to plot the curve $x^4 + x^2y^2 - y^2 = 0$ for $x$ and $y$ both between $-1$ and $+1$.

(a) Sampling this curve as given in rectangular coordinates, how many samples do we need to take from the curve to obtain a nice plot?

(b) Convert the curve into polar coordinates and plot. Give all commands used to obtain the plot.
How many samples of the curve are needed here?

9. Solve $x^2a^2 - 2x^2a - 3x^2 - xa^2 + 4xa - 3x + a^2 + 2a - 15$ for $x$ for all values of the parameter $a$.
Be as complete as possible in your description of the solution.

10. Find the point with real coordinates on the curve $xy - 2x + 3$ which lies closest to the origin.

11. How many real solutions does the system $\begin{cases} x^2 - 2y^2 - 1 = 0 \\ xy - 2x - 3 = 0 \end{cases}$ have?

12. Consider $y'' + 6y' + 13y = 0$, with $y(\pi/2) = -2$ and $y'(\pi/2) = 8$.

(a) Find an exact solution to this initial value problem and use this to create a function `s` which returns a numerical 10-digit floating-point approximation of the solution.

(b) Solve this initial value problem numerically. Compare the solution with the value for $y(2)$ and also with `s`(2) obtained in (a).

13. Create a 5-by-5 Vandermonde matrix where the $(i, j)$-th entry is $x_i^{j-1}$. Show that its determinant equals $(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_1 - x_5)(x_2 - x_3)(x_2 - x_4)(x_2 - x_5)(x_3 - x_4)(x_3 - x_5)(x_4 - x_5)$.

# Review of MATLAB

In the fifth and last part of the course, nine lectures introduces us to MATLAB. Below is a list of questions to review for the final exam. Consider also the quizzes, homework assignments, and project.

1. Explain the difference between the `*` and `.*` operators.
   Give an example where the `*` must be used instead of the `.*` operator.
   Give an example where the `.*` must be used instead of the `*` operator.

2. Fermat's spiral is defined in polar coordinates by $r = \sqrt{t}$.
   Give the MATLAB commands to make a plot for $t$ from 0 to 25.

3. Give the MATLAB commands to make a plot of the surface $z = y^2 \cos(x + y)$,
   for $x$ and $y$ ranging between $-\pi$ and $+\pi$.

4. The twisted cubic is a space curve, which can be defined in two ways:

   (a) as $(t, t^2, t^3)$ in parametric representation, with parameter $t$; and

   (b) as the intersection of two surfaces: $y - x^2 = 0$ and $z - x^3 = 0$.

   Give all MATLAB commands to make a figure of the twisted cubic, with two subplots in the same figure window. The first subplot must use the first parametric definition, while the second subplot should show the two surfaces whose intersection defines the twisted cubic. The region of interest in which we wish to see the twisted cubic is the unit cube $[-1, 1] \times [-1, 1] \times [-1, 1]$.

5. In MATLAB we have seen three different situations where we encountered approximate data. For each such different situation, we processed the data differently. Describe briefly each situation and list the most important MATLAB command that was used in processing the approximate data.

6. We encountered the commands `polyfit` and `spline` in MATLAB.

   (a) Explain the difference between the commands `polyfit` and `spline`.

   (b) Describe a problem for which you should use `polyfit` rather than `spline`.

   (c) Describe a problem for which you should use `spline` rather than `polyfit`.

7. Simpson's rule to approximate the definite integral of $f(x)$ over $[a, b]$ is defined as

$$\int_a^b f(x)dx \approx \frac{b-a}{3}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right).$$

   (a) Write an m-file to implement the function

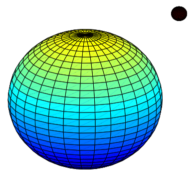
```
function y = simpson(f,a,b)
% returns an approximation for the integral of f(x) over [a,b]
% using Simpson's rule
```

   (b) Use simpson to approximate the integral of $\cos(x)$ for $x$ between $\pi/4$ and $\pi/2$.

8. Suppose we want to sample $f(t) = \sin(2\pi 10t)$. How many samples of $f(t)$ should we take in the interval $[0, 3]$ for a good plot of $f(t)$? What if instead of 10 we have any positive number $k$?

9. How would you in MATLAB define a permutation matrix $P$ so that `P * [1 2 3 .. n]'` returns `[n .. 3 2 1]'`, for some number `n`? (*Hint*: take `n` = 3 and generalize.)

10. Give all MATLAB commands to make the following plot:



Start with a unit sphere as the big planet. The small sphere is ten times smaller as the big planet. Lift the satellite two units up and shift the second coordinate with negative two.

11. Give **all** MATLAB commands to solve the following problem:

$$\max_{x,y} 23x + 44y$$

$$\text{subject to} \left\{ \begin{array}{rcl} x + 12y & \leq & 123 \\ 24x + 30y & \leq & 912 \\ -2x + 83y & \leq & 134 \\ & x \geq 0, y \geq 0 & \end{array} \right.$$

How do you verify for which constraints the equality is reached?

**FINAL EXAM is at the usual location (SEL 2249) on Tuesday, December 7, 2004, from 1:00 to 3:00PM**

In case of a scheduling conflict with another final exam, please let me know as soon as possible. Since a separate computer lab reservation is required to hold a makeup, it is strongly advised that you give preference to MCS 320 when resolving the conflict.

Observe the university rules concerning incompletes. An incomplete can only be granted if all of the following conditions are satisfied:

1. The student is in good standing and needs only a final exam to complete the course. In particular, this means that no midterms are skipped, attendance to the discussion sessions was documented by quiz scores, and all projects received a satisfactory grade.
2. Some event (for which adequate documentation can be provided) prevented the student from doing a makeup final exam.

Note that these rules are from the university, and that the administration needs to approve incompletes.