## Associativity of addition of natural numbers from Peano axioms

We want to prove the following statement:

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\begin{equation*}
a, b, c \in \mathbf{Z}^{+} \Rightarrow(a+b)+c=a+(b+c) \tag{1}
\end{equation*}
$$

We shall use the Peano definition of the set of natural numbers i.e. defined as a set $\mathbf{N}$ with a distinguished element $1 \in \mathbf{N}$ and an injective map $s: \mathbf{N} \rightarrow \mathbf{N}$ such that 1 does not belong to $s(\mathbf{N})$ and such that the following induction axiom (ind) is satisfied:
(ind) If $A \subseteq \mathbf{N}$ such that $1 \in \mathbf{N}$ and $a \in A \Rightarrow s(a) \in A$ then $A=\mathbf{N}$.
In such a set $\mathbf{N}$ one defines addition of $b \in \mathbf{N}$ to $a \in \mathbf{N}$ so that the following two properties are satisfied:
(i) $a+1=s(a)$
(ii) If $a+b$ is defined, then $a+(b+1)$ is defined as $s(a+b)$.

Note that the set $B$ of $b \in \mathbf{N}$ for which $a+b$ is defined contains 1 and by (ii) if $b \in B$ then $b+1 \in B$. Therefore, by (ind) $B=\mathbf{N}$ and hence $a+b$ is defined for all $a, b \in \mathbf{N}$.

We shall prove the identity (1) using induction over $c$ i.e. we show that the set $C$ defined as subset of $\mathbf{N}$ consisting of $c \in \mathbf{N}$ for which (1) is true is equal to the set $\mathbf{N}$ (this is the same as to say that (1) is true for any $c \in \mathbf{N}$ ).

Claim 1: $1 \in C$ i.e. associativity takes place for $c=1$ i.e.

$$
(a+b)+1=a+(b+1)
$$

Indeed, LHS by part (i) of definition of addition is $s(a+b)$ and RHS is $s(a+b)$ by part (ii). Hence both sides are the values of function $s$ at $a+b$ and hence are equal (by definition of a function).

Claim 2: if $(a+b)+c=a+(b+c)$ then $(a+b)+(c+1)=a+(b+(c+1))$.
. Indeed

$$
\begin{gathered}
(a+b)+(c+1)=s((a+b)+c) \quad \text { (by (ii) in definition of addition) } \\
=s(a+(b+c)) \quad(\text { by assumption of Claim } 2) \\
=a+((b+c)+1) \quad(\text { by (ii) of definition of addition) }
\end{gathered}
$$

$=a+(b+(c+1)) \quad$ (by Claim 1 i.e. associativity of addition when last summand equals 1 )
QED.

