Associativity of addition of natural numbers from Peano axioms

We want to prove the following statement:

$$a, b, c \in \mathbf{Z}^+ \Rightarrow (a+b) + c = a + (b+c)$$
 (1)

We shall use the Peano definition of the set of natural numbers i.e. defined as a set \mathbf{N} with a distinguished element $1 \in \mathbf{N}$ and an injective map $s : \mathbf{N} \to \mathbf{N}$ such that 1 does not belong to $s(\mathbf{N})$ and such that the following induction axiom (ind) is satisfied:

(ind) If $A \subseteq \mathbf{N}$ such that $1 \in \mathbf{N}$ and $a \in A \Rightarrow s(a) \in A$ then $A = \mathbf{N}$.

In such a set **N** one defines addition of $b \in \mathbf{N}$ to $a \in \mathbf{N}$ so that the following two properties are satisfied:

(i) a + 1 = s(a)(ii) If a + b is defined, then a + (b + 1) is defined as s(a + b).

Note that the set B of $b \in \mathbb{N}$ for which a + b is defined contains 1 and by (ii) if $b \in B$ then $b + 1 \in B$. Therefore, by (ind) $B = \mathbb{N}$ and hence a + b is defined for all $a, b \in \mathbb{N}$.

We shall prove the identity (1) using induction over c i.e. we show that the set C defined as subset of **N** consisting of $c \in \mathbf{N}$ for which (1) is true is equal to the set **N** (this is the same as to say that (1) is true for any $c \in \mathbf{N}$).

Claim 1: $1 \in C$ i.e. associativity takes place for c = 1 i.e.

$$(a+b) + 1 = a + (b+1)$$

Indeed, LHS by part (i) of definition of addition is s(a+b) and RHS is s(a+b) by part (ii). Hence both sides are the values of function s at a + b and hence are equal (by definition of a function).

Claim 2: if
$$(a + b) + c = a + (b + c)$$
 then $(a + b) + (c + 1) = a + (b + (c + 1))$.

. Indeed

$$(a+b) + (c+1) = s((a+b) + c)$$
 (by (ii) in definition of addition)

= s(a + (b + c)) (by assumption of Claim 2)

= a + ((b + c) + 1) (by (ii) of definition of addition)

= a + (b + (c+1)) (by Claim 1 i.e. associativity of addition when last summand equals 1) QED.