GEORG CANTOR AND HIS HERITAGE[1]

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God is no geometer, rather an unpredictable poet.
(Geometers can be unpredictable poets, so there could be room for compromise.)

V. Tasić [T], on the XIXth century romanticism

Introduction

Georg Cantor's grand meta-narrative, Set Theory, created by him almost single-handedly in the span of about fifteen years, resembles a piece of high art more than a scientific theory.

Using a slightly modernized language, basic results of set theory can be stated in a few lines.

Consider the category of all sets with arbitrary maps as morphisms. Isomorphism classes of sets are called *cardinals*. Cardinals are well–ordered by the sub–object relation, and the cardinal of the set of all subsets of U is strictly larger than that of U (this is of course proved by the famous diagonal argument).

This motivates introduction of another category, that of well ordered sets and monotone maps as morphisms. Isomorphism classes of these are called *ordinals*. They are well–ordered as well. The Continuum Hypothesis is a guess about the order structure of the initial segment of cardinals.

Thus, exquisite minimalism of expressive means is used by Cantor to achieve a sublime goal: understanding infinity, or rather infinity of infinities. A built–in self–referentiality and the forceful extension of the domain of mathematical intuition (principles for building up new sets) add to this impression of combined artistic violence and self–restraint.

Cantor himself would have furiously opposed this view. For him, the discovery of the hierarchy of infinities was a revelation of God–inspired Truth.

But mathematics of the XXth century reacted to Cantor's oeuvre in many ways that can be better understood in the general background of various currents of contemporary science, philosophical thought, and art.

Somewhat provocatively, one can render one of Cantor's principal insights as follows:

$$2^x \textit{ is considerably larger than } x.$$

[1] Talk at the meeting of the German Mathematical Society and the Cantor Medal award ceremony.

Here x can be understood as an integer, an arbitrary ordinal, or a set; in the latter case 2^x denotes the set of all subsets of x. Deep mathematics starts when we try to make this statement more precise and to see *how much larger* 2^x is.

If x is the first infinite ordinal, this is the Continuum Problem.

I will argue that properly stated for finite x, this question becomes closely related to a universal NP problem.

I will then discuss assorted topics related to the role of set theory in contemporary mathematics and the reception of Cantor's ideas.

Axiom of Choice and P/NP–problem *or* finite as poor man's transfinite[2]

In 1900, at his talk at the second ICM in Paris, Hilbert put the Continuum Hypothesis at the head of his list of 23 outstanding mathematical problems. This was one of the highlights of Cantor's scientific life. Cantor invested much effort consolidating the German and international mathematical community into a coherent body capable of counterbalancing a group of influential professors opposing set theory.

The opposition to his theory of infinity, however, continued and was very disturbing to him, because the validity of Cantor's new mathematics was questioned.

In 1904, at the next International Congress, König presented a talk in which he purported to show that continuum could not be well–ordered, and therefore the Continuum Hypothesis was meaningless.

Dauben writes ([D], p. 283): "The dramatic events of König's paper read during the Third International Congress for Mathematicians in Heidelberg greatly upset him [Cantor]. He was there with his two daughters, Else and Anna–Marie, and was outraged at the humiliation he felt he had been made to suffer."

It turned out that König's paper contained a mistake; Zermelo soon afterwards produced a proof that any set could be well ordered using his then brand new Axiom of Choice. This axiom essentially postulates that, starting with a set U, one can form a new set, whose elements are pairs (V, v) where V runs over all non–empty subsets of U, and v is an element of V.

A hundred years later, the mathematical community did not come up with a compendium of new problems for the coming century similar to the Hilbert's list. Perhaps, the general vision of mathematics changed — already in Hilbert's list a considerable number of items could be better described as research programs rather

[2] I am not hinting at the Clay USD 10^6 Prize for a solution of the P/NP–problem.

than well–defined problems, and this seems to be a more realistic way of perceiving our work in progress.

Still, a few clearly stated important questions remain unanswered, and recently seven such questions were singled out and endowed with a price tag. Below I will invoke one of these questions, the P/NP problem and look at it as a finitary travesty of Zermelo's Axiom of Choice.

Let $U_m = \mathbf{Z}_2^m$ be the set of m–bit sequences. A convenient way to encode its subsets is via Boolean polynomials. Using the standard – and more general – language of commutative algebra, we can identify each subset of U_m with the 0–level of a unique function $f \in B_m$ where we define the algebra of Boolean polynomials as

$$B_m := \mathbf{Z}_2[x_1, \dots, x_m]/(x_1^2 + x_1, \dots, x_m^2 + x_m).$$

Hence Zermelo's problem – *choose an element in each non–empty subset of* U – translates into: *for each Boolean polynomial, find a point at which the polynomial equals 0, or prove that the polynomial is identically 1.* Moreover, we want to solve this problem in time bounded by a polynomial of the bit size of the code of f.

This leads to a universal ("maximally difficult") NP–problem if one writes Boolean polynomials in the following version of disjunctive normal form. Code of such a form is a family

$$u = \{m; (S_1, T_1), \dots, (S_N, T_N)\},\ \ m \in \mathbf{N};\ S_i, T_i \subset \{1, \dots, m\}.$$

The bit size of u is mN, and the respective Boolean polynomial is

$$f_u := 1 + \prod_{i=1}^{N} \left(1 + \prod_{k \in S_i} (1 + x_k) \prod_{j \in T_i} x_j \right)$$

This encoding provides for a fast check of the inclusion relation for the elements of the respective level set. The price is that the uniqueness of the representation of f is lost, and moreover, the identity $f_u = f_v (?)$ becomes a computationally hard problem.

In particular, even the following weakening of the finite Zermelo problem becomes NP–complete and hence currently intractable: *check whether a Boolean polynomial given in disjunctive normal form is non–constant.*

Zermelo's Axiom of Choice aroused a lively international discussion published in the first issue of Mathematische Annalen of 1905. A considerable part of it was focused on the psychology of mathematical imagination and on the reliability of its fruits. Baffling questions of the type: "How can we be sure that during the course of a proof we keep thinking about the same set?" kept emerging. If we imagine that at least a part of computations that our brains perform can be adequately modeled by finite automata, then quantitative estimates of the required resources as provided by the theory of polynomial time computability might eventually be of use in neuroscience and by implication in psychology.

A recent paper in "Science" thus summarizes some experimental results throwing light on the nature of mental representation of mathematical objects and physiological roots of divergences between, say, intuitionists and formalists:

"[...] our results provide grounds for reconciling the divergent introspection of mathematicians by showing that even within the small domain of elementary arithmetic, multiple mental representations are used for different tasks. Exact arithmetic puts emphasis on language–specific representations and relies on a left inferior frontal circuit also used for generating associations between words. Symbolic arithmetic is a cultural invention specific to humans, and its development depended on the progressive improvement of number notation systems. [...]

Approximate arithmetic, in contrast, shows no dependence on language and relies primarily on a quantity representation implemented in visuo–spatial networks of the left and right parietal lobes." ([DeSPST], p. 973).

In the next section, I will discuss an approach to the Continuum Conjecture which is clearly inspired by the domination of the visuo–spatial networks, and conjecturally better understood in terms of probabilistic models than logic or Boolean automata.

Appendix: some definitions. For completeness, I will remind the reader of the basic definitions related to the P/NP problem. Start with an infinite constructive world U in the sense of [Man], e.g. natural numbers $\mathbf{N}$. A subset $E\subset U$ *belongs to the class* P if it is decidable and the values of its characteristic function χ_E are computable in polynomial time on all arguments $x\in E$.

Furthermore, $E\subset U$ *belongs to the class* NP if it is a polynomially truncated projection of some $E'\subset U\times U$ belonging to P: for some polynomial G,

$$u\in E\ \Leftrightarrow\ \exists\,(u,v)\in E' \text{ with } |v|\le G(|u|)$$

where $|v|$ is the bit size of v. In particular, $P\subset NP$.

Intuitively, $E\in NP$ means that for each $u\in E$ *there exists a polynomially bounded proof* of this inclusion (namely, the calculation of $\chi_{E'}(u,v)$ for an appropriate v). However, to find such a proof (i. e. v) via the naive search among all v with $|v|\le G(|u|)$ can take exponential time.

The set $E\subset U$ is called *NP–complete* if, for any other set $D\subset V, D\in NP$, there exists a polynomial time computable function $f: V\to U$ such that $D=f^{-1}(E)$, that is, $\chi_D(v)=\chi_E(f(v))$.

The encoding of Boolean polynomials used above is explained and motivated by the proof of the NP–completeness: see e. g. [GaJ], sec. 2.6.

The Continuum Hypothesis and random variables

Mumford in [Mum], p. 208, recalls an argument of Ch. Freiling ([F]) purporting to show that Continuum Hypothesis is "obviously" false by considering the following situation:

"Two dart players independently throw darts at a dartboard. If the continuum hypothesis is true, the points P on the surface of a dartboard can be well ordered so that for every P, the set of Q such that $Q < P$, call it S_Q, is countable. Let players 1 and 2 hit the dart board at points P_1 and P_2. Either $P_1 < P_2$ or $P_2 < P_1$. Assume the first holds. Then P_1 belongs to a countable subset S_{P_2} of the points on on the dartboard. As the two throws were independent, we may treat throw 2 as taking place first, then throw 1. After throw 2, this countable set S_{P_2} has been fixed. But every countable set is measurable and has measure 0. The same argument shows that the probability of P_2 landing on S_{P_1} is 0. Thus almost surely neither happened and this contradicts the assumption that the dartboard is the first uncountable cardinal! [...]

I believe [...] his 'proof' shows that if we make random variables one of the basic elements of mathematics, it follows that the C.H. is false and we will get rid of one of the meaningless conundrums of set theory."[3]

Freiling's work was actually preceded by that of Scott and Solovay, who recast in terms of "logically random sets" P. Cohen's forcing method for proving the consistency of the negated CH with Zermelo–Frenkel axioms. Their work has shown that one can indeed put random variables in the list of basic notions and use them in a highly non–trivial way.

P. Cohen himself ended his book by suggesting that the view that CH is "obviously false" may become universally accepted.

However, whereas the Scott–Solovay reasoning proves a precise theorem about the formal language of the Set Theory, Freiling's argument appeals directly to our physical intuition, and is best classified as a *thought experiment.* It is similar in nature to some classical thought experiments in physics, deducing e.g. various dynamic consequences from the impossibility of *perpetuum mobile.*

The idea of thought experiment, as opposed to that of logical deduction, can be generally considered as a right–brain equivalent of the left brain elementary logical

[3] Mumford's paper is significantly called *"The dawning of the age of stochasticity."* David assured me that he had no intention to refer to Giambattista Vico's theory of historical cycles which in the rendering of H. Bloom ([B]) sounds thus: " Giambattista Vico, in his *New Science,* posited a cycle of three phases — Theocratic, Aristocratic, Democratic — followed by a chaos of which a New Theocratic Age would at last emerge."

operations. A similar role is played by good metaphors. When we are comparing the respective capabilities of two brains, we are struck by what I called elsewhere 'the inborn weakness of metaphors': they resist becoming building blocks of a system. One can only more or less artfully put one metaphor upon another, and the building will stand or crumble upon its own weight independently of its truth or otherwise.

Physics disciplines thought experiments as poetry disciplines metaphors, but only logic has an inner discipline.

Successful thought experiments produce mathematical truths which, after being accepted, solidify into axioms, and the latter start working on the treadmill of logical deductions.

Foundations and Physics

I will start this section with a brief discussion of the impact of Set Theory on the foundations of mathematics. I will understand "foundations" neither as the para–philosophical preoccupation with nature, accessibility, and reliability of mathematical truth, nor as a set of normative prescriptions like those advocated by finitists or formalists.

I will use this word in a loose sense as a general term for the historically variable conglomerate of rules and principles used to organize the already existing and always being created anew body of mathematical knowledge of the relevant epoch. At times, it becomes codified in the form of an authoritative mathematical text as exemplified by Euclid's Elements. In another epoch, it is better expressed by nervous self–questioning about the meaning of infinitesimals or the precise relationship between real numbers and points of the Euclidean line, or else, the nature of algorithms. In all cases, foundations in this wide sense is something which is of relevance to a working mathematician, which refers to some basic principles of his/her trade, but which does not constitute the essence of his/her work.

In the XXth century, all of the main foundational trends referred to Cantor's language and intuition of sets.

The well developed project of Bourbaki gave a polished form to the notion that *any* mathematical object $\mathcal{X}$ (group, topological space, integral, formal language ...) could be thought of as a set X with an additional structure x. This notion had emerged in many specialized research projects, from Hilbert's *Grundlagen der Geometrie* to Kolmogorov's identification of probability theory with measure theory.

The additional structure x in $\mathcal{X} = (X, x)$ is an element of another set Y belonging to the *échelle* constructed from X by standard operations and satisfying conditions (axioms) which are also formulated entirely in terms of set theory. Moreover, the nature of elements of X is inessential: a bijection $X \to X'$ mapping x to x' produces an isomorphic object $\mathcal{X}' = (X', x')$. This idea played a powerful unifying and clarifying role in mathematics and led to spectacular developments far outside the Bourbaki group. Insofar as it is accepted in thousands of research papers, one can simply say that the language of mathematics is the language of set theory.

Since the latter is so easily formalized, this allowed logicists to defend the position that their normative principles should be applied to all of mathematics and to overstate the role of "paradoxes of infinity" and Gödel's incompleteness results.

However, this fact also made possible such self–reflexive acts as inclusion of metamathematics into mathematics, in the form of model theory. Model theory studies special algebraic structures – formal languages – considered in turn as mathematical objects (structured sets with composition laws, marked elements etc.), and their interpretations in sets. Baffling discoveries such as Gödel's incompleteness of arithmetics lose some of their mystery once one comes to understand their content as a statement that a certain algebraic structure simply is not finitely generated with respect to the allowed composition laws.

When at the next stage of this historical development, sets gave way to categories, this was at first only a shift of stress upon morphisms (in particular, isomorphisms) of structures, rather than on structures themselves. And, after all, a (small) category could itself be considered as a set with structure. However, primarily thanks to the work of Grothendieck and his school on the foundations of algebraic geometry, categories moved to the foreground. Here is an incomplete list of changes in our understanding of mathematical objects brought about by the language of categories. Let us recall that generally objects of a category C are not sets themselves; their nature is not specified; only morphisms $Hom_C(X,Y)$ are sets.

A. An object X of the category C can be identified with the functor it represents: $Y \mapsto Hom_C(Y,X)$. Thus, if C is small, initially structureless X becomes a structured set. This external, "sociological" characterization of a mathematical object defining it through its interaction with all objects of the same category rather than in terms of its intrinsic structure, proved to be extremely useful in all problems involving e. g. moduli spaces in algebraic geometry.

B. Since two mathematical objects, if they are isomorphic, have exactly the same properties, it does not matter how many pairwise isomorphic objects are contained in a given category C. Informally, if C and D have "the same" classes of isomorphic objects and morphisms between their representatives, they should be considered as equivalent. For example, the category of "all" finite sets is equivalent to any category of finite sets in which there is exactly one set of each cardinality 0,1,2,3, $\dots$.

This "openness" of a category considered up to equivalence is an essential trait, for example, in the abstract computability theory. Church's thesis can be best understood as a postulate that there is an open category of "constructive worlds" — finite or countable structured sets and computable morphisms between them —

such that any infinite object in it is isomorphic to the world of natural numbers, and morphisms correspond to recursive functions (cf. [Man] for more details). There are many more interesting infinite constructive worlds determined by widely diverging internal structures: words in a given alphabet, finite graphs, Turing machines, etc. However, they are all isomorphic to $\mathbf{N}$ due to the existence of computable numerations.

C. The previous remark also places limits on the naive view that categories "are" special structured sets. In fact, if it is natural to identify categories related by an equivalence (not necessarily bijective on objects) rather than isomorphism, then this view becomes utterly misleading.

More precisely, what happens is the slow emergence of the following hierarchical picture. Categories themselves form objects of a larger category Cat morphisms in which are functors, or "natural constructions" like a (co)homology theory of topological spaces. However, functors do not form simply a set or a class: they also form objects of a category. Axiomatizing this situation we get a notion of *2–category* whose prototype is Cat. Treating 2–categories in the same way, we get 3–categories etc.

The following view of mathematical objects is encoded in this hierarchy: there is no equality of mathematical objects, only equivalences. And since an equivalence is also a mathematical object, there is no equality between them, only the next order equivalence etc., *ad infinitim.*

This vision, due initially to Grothendieck, extends the boundaries of classical mathematics, especially algebraic geometry, and exactly in those developments where it interacts with modern theoretical physics.

With the advent of categories, the mathematical community was cured of its fear of classes (as opposed to sets) and generally "very large" collections of objects.

In the same vein, it turned out that there are meaningful ways of thinking about "all" objects of a given kind, and to use self–reference creatively instead of banning it completely. This is a development of the old distinction between sets and classes, admitting that at each stage we get a structure similar to but not identical with the ones we studied at the previous stage.

In my view, Cantor's prophetic vision was enriched and not shattered by these new developments.

What made it recede to the background, together with preoccupations with paradoxes of infinity and intuitionistic neuroses, was a renewed interaction with physics and the transfiguration of formal logic into computer science.

The birth of quantum physics radically changed our notions about relationships between reality, its theoretical descriptions, and our perceptions. It made clear that Cantor's famous definition of sets ([C]) represented only a distilled classical mental view of the material world as consisting of pairwise distinct things residing in space:

"Unter einer 'Menge' verstehen wir jede Zusammenfassung M von bestimmten wohlunterschiedenen Objekten m unserer Anschauung oder unseres Denkens (welche die 'Elemente' von M genannt werden) zu einem Ganzen."

"By a 'set' we mean any collection M into a whole of definite, distinct objects m (called the 'elements' of M) of our perception or our thought."

Once this view was shown to be only an approximation to the incomparably more sophisticated quantum description, sets lost their direct roots in reality. In fact, the structured sets of modern mathematics used most effectively in modern physics are not sets of things, but rather sets of *possibilities.* For example, the phase space of a classical mechanical system consists of pairs *(coordinate, momentum)* describing all possible states of the system, whereas after quantization it is replaced by the space of complex probability amplitudes: the Hilbert space of L_2–functions of the coordinates or something along these lines. The amplitudes are all possible quantum superpositions of all possible classical states. It is a far cry from a set of things.

Moreover, requirements of quantum physics considerably heightened the degree of tolerance of mathematicians to the imprecise but highly stimulating theoretical discourse conducted by physicists. This led, in particular, to the emergence of Feynman's path integral as one of the most active areas of research in topology and algebraic geometry, even though the mathematical status of path integral is in no better shape than that of Riemann integral before Kepler's "Stereometry of Wine Barrels".

Computer science added a much needed touch of practical relevance to the essentially hygienic prescriptions of formal logic. The introduction of the notion of "success with high probability" into the study of algorithmic solvability helped to further demolish mental barriers which fenced off foundations of mathematics from mathematics itself.

Appendix: Cantor and physics. It would be interesting to study Cantor's natural philosophy in more detail. According to [D], he directly referred to possible physical applications of his theory several times.

For example, he proved that that if one deletes from a domain in $\mathbf{R}^n$ any dense countable subset (e.g. all algebraic points), then any two points of the complement can be connected by a continuous curve. His interpretation: continuous motion is possible even in discontinuous spaces, so "our" space might be discontinuous itself, because the idea of its continuity is based upon observations of continuous motion. Thus a revised mechanics should be considered.

At a meeting of GDNA in Freiburg, 1883, Cantor said: "One of the most important problems of set theory [...] consists of the challenge to determine the various valences or powers of sets present in all of Nature in so far as we can know them" ([D], p. 291).

Seemingly, Cantor wanted atoms (monads) to be actual points, extensionless, and in infinite number in Nature. "Corporeal monads" (massive particles? Yu.M.) should exist in countable quantity. "Aetherial monads" (massless quanta? Yu.M.) should have had cardinality aleph one.

Coda: Mathematics and postmodern condition

Already during Cantor's life time, the reception of his ideas was more like that of new trends in the art, such as impressionism or atonality, than that of new scientific theories. It was highly emotionally charged and ranged from total dismissal (Kronecker's "corrupter of youth") to highest praise (Hilbert's defense of "Cantor's Paradise"). (Notice however the commonly overlooked nuances of both statements which subtly undermine their ardor: Kronecker implicitly likens Cantor to Socrates, whereas Hilbert with faint mockery hints at Cantor's conviction that Set Theory is inspired by God.)

If one accepts the view that Bourbaki's vast construction was the direct descendant of Cantor's work, it comes as no surprise that it shared the same fate: see [Mas]. Especially vehement was reaction against "new maths": an attempt to reform the mathematical education by stressing precise definitions, logic, and set theoretic language rather than mathematical facts, pictures, examples and surprises.

One is tempted to consider this reaction in the light of Lyotard's ([L]) famous definition of the postmodern condition as "incredulity toward meta–narratives" and Tasić's remark that mathematical truth belongs to "the most stubborn meta–narratives of Western culture" ([T], p. 176).

In this stubbornness lies our hope.

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Appendix: Chronology of Cantor's life and mathematics

(following [PI] and [D])

March 3, 1845: Born in St Petersburg, Russia.

1856: Family moves to Wiesbaden, Germany.

1862 – 1867: Cantor studies in Zürich, Berlin, Göttingen and again Berlin.

1867 – 1869: First publications in number theory, quadratic forms.

1869: Habilitation in the Halle University.

1870 – 1872: Works on convergence of trigonometric series.

1872 – 1879: Existence of different magnitudes of infinity, bijections $\mathbf{R} \to \mathbf{R}^n$, studies of relations between continuity and dimension.

November 29, 1873: Cantor asked in a letter to Dedekind whether there might exist a bijection between $\mathbf{N}$ and $\mathbf{R}$ ([D], p.49). Shortly after Christmas he found his diagonal procedure ([D], p. 51 etc.).

1874: First publication on set theory.

1879 – 1884: Publication of the series *Über unendliche lineare Punktmannichfaltigkeiten.*

1883: *Grundlagen einer allgemeinen Mannigfaltigkeitslehre. Ein mathematisch–philosophischer Versuch in der Lehre des Unendlichen.*

May 1884: First nervous breakdown, after a successful and enjoyable trip to Paris: depression lasting till Fall: [D]. p. 282.

1884–85: Contact with Catholic theologians, encouragement from them, but isolation in Halle. [D], p. 146: "... early in 1885, Mittag–Leffler seemed to close the last door on Cantor's hopes for understanding and encouragement among mathematicians."

September 18, 1890: Foundation of the German Mathematical Society; Cantor becomes its first President.

1891: Kronecker died.

1895 – 1895: *"Beiträge zur Begründung der transfiniten Mengenlehre"* (Cantor's last major mathematical publication).

1897: The first ICM. Set theory is very visible.

1897: "Burali–Forti [...] was the first mathematician to make public the paradoxes of transfinite set theory" ([D]). He argued that *all* ordinals, if any pair of them is comparable, would form an Ordinal which is greater than itself. He concluded that not all ordinals are comparable. Cantor instead believed that all ordinals do not form an ordinal, just as all sets do not form a set.

1899: Hospitalizations in Halle Nervenklinik before and after the death of son Rudolph.

1902–1903, winter term: Hospitalization.

Oct. 1907 – June 1908: Hospitalization.

Sept. 1911 – June 1912: Hospitalization.

1915: Celebration of the 70th anniversary of Cantor's birth, on a national level because of WW I.

May 1917 – Jan. 6, 1918: Hospitalization; Cantor dies at Halle Klinik.

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