NUMERICAL CHARACTERISTICS OF SYSTEMS OF STRAIGHT LINES ON COMPLETE INTERSECTIONS

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We obtain an equation for the number of straight lines on the complete intersection of hypersurfaces and find Hilbert's polynomial for the variety of straight lines of a cubic three-dimensional hypersurface.

Let $V_{r-s}^{n_1,\ldots,n_s}$ be the algebraic variety in the projective space P^r of dimension r, which is the complete intersection of s hypersurfaces of degrees $n_1, \ldots, n_s$. We know that straight lines in P^r, lying on $V_{r-s}^{n_1,\ldots,n_s}$, if in general they exist, can be parameterized by the algebraic variety $s_1(V_{r-s}^{n_1,\ldots,n_s})$. This variety is canonically embedded in a Grassman variety and so in a projective space.

It follows from the equation obtained by Predonzan that for a general $V_{r-s}^{n_1,\ldots,n_s}$ we have

$$\dim s_1(V_{r-s}^{n_1,\ldots,n_s}) = 2(r-1) - \sum_{i=1}^{s}(n_i+1).$$

Hence, if

$$2(r-1) - \Sigma(n_i+1) = 0, \tag{1}$$

then there is a finite number of straight lines on a general $V_{r-s}^{n_1,\ldots,n_s}$. For example, it has long been known that there are 27 straight lines on a nonsingular cubic surface.

We find the number of straight lines on a general $V_{r-s}^{n_1,\ldots,n_s}$ in the case (1), i.e., when their number is finite.

For P^α and P^β, embedded in P^γ (we can assume that $\alpha+\beta>\gamma$), and not contained in any $P^{\gamma-1}$, the number of straight lines in P^γ intersecting them and lying on a general $V_{\gamma-1}^n$ will be denoted by $N_\gamma^{\alpha,\beta}$.

The fundamental result is

THEOREM 1. If $\alpha+\beta = n+1$, then $N_\gamma^{\alpha,\beta}$ is finite and equal to

$$n\cdot n!\left[\sigma_{\alpha-1}\left(\ldots,\frac{n-i}{i},\ldots\right) - \sigma_{n-\gamma}\left(\ldots,\frac{n-i}{i},\ldots\right)\right],$$

where σ_k is the k-th elementary symmetric polynomial in the arguments $\frac{n-1}{1}, \frac{n-2}{2}, \ldots, \frac{n-i}{i}, \ldots, \frac{1}{n-1}$.

From this there easily follows

THEOREM 2. If $2r-n-3=0$, the number of straight lines on V_{r-1}^n is finite and equal to

$$n\cdot n!\left[\sigma_{\frac{n-1}{2}}\left(\frac{n-1}{1},\ldots,\frac{n-i}{i},\ldots,\frac{1}{n-1}\right) - \sigma_{\frac{n-3}{2}}\left(\frac{n-1}{1},\ldots,\frac{n-i}{i},\ldots,\frac{1}{n-1}\right)\right],$$

where the notation is the same as in Theorem 1.

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Let $\Omega_{a_0 a_1}$ $(0 \leqslant a_0 < a_1 \leqslant r)$ denote the Schubert variety of straight lines of the space P^r contained in P^{a_1} and intersecting $P^{a_0} \subset P^{a_1}$. We know that this is an irreducible variety of dimension $a_0 + a_1 - 1$ and if the numbers $r, n_1, \ldots, n_s$ satisfy Eq. (1), the number

$$K_{i_1, \ldots, i_s} = (\Omega_{i_1, 2r-n_1-2-i_1} \ldots \Omega_{i_s, 2r-n_s-2-i_s})$$

is defined for any set of numbers $i_1, \ldots, i_s$ satisfying the inequalities

$$\max(r-n_1-2, 0) < i_1 < r - \frac{n_1}{2} - 1, \ldots, \max(r-n_s-2, 0) < i_s < r - \frac{n_s}{2} - 1,$$

calculated in a standard manner using the equations for multiplication in a ring of classes of cycles to within numerical equivalence of a Grassman variety.

THEOREM 3. The number of straight lines on $V_{r-s}^{n_1 \ldots n_s}$ in the case when

$$2(r-1) - \sum_{i=1}^{s} (n_i + 1) = 0$$

is finite and equal to

$$\sum_{i_1 \ldots i_s} N_{r-i_1}^{n_1+2+i_1-r, r-i_1} \ldots N_{r-i_s}^{n_s+2+i_s-r, r-i_s} K_{i_1 \ldots i_s},$$

where each of the indices $i_1, \ldots, i_s$ runs through the appropriate set of values:

$$\max(0, r-n_1-2) \leqslant i < r - \frac{n_1}{2} - 1,$$

$$\max(0, r-n_s-2) \leqslant i_s < r - \frac{n_s}{2} - 1.$$

For example,

$$\# s_1(V_3^5) = 5^3 \cdot 23, \ \# s_1(V_3^{3,3}) = 3^4 \cdot 13, \ \# s_1(V_3^{2,4}) = 2^7 \cdot 5.$$

In the second section we calculate the Hilbert polynomial of the variety of straight lines lying on a cubic hypersurface in P. It is $\frac{45}{2} n^2 - \frac{45}{2} n + 6$. Some of its coefficients were calculated in a paper by Fano.

§1. Finite Systems of Straight Lines. For P^α and P^β, embedded in P^γ $(\alpha + \beta > \gamma)$ and not contained in $P^{\gamma-1}$ (or contained in $P^{\gamma-1}$, but not contained in any $P^{\gamma-2}$), the variety of straight lines intersecting them will be denoted by $M_\gamma^{\alpha,\beta}$ (or $\bar{M}_\gamma^{\alpha,\beta}$).

The variety of straight lines of $M_\gamma^{\alpha,\beta}$ (or $\bar{M}_\gamma^{\alpha,\beta}$) lying on $V_{\gamma-1}^n$ will be denoted by $L_\gamma^{\alpha,\beta}$ (or $\bar{L}_\gamma^{\alpha,\beta}$). $P^{N(n,\gamma)}$ denotes the space of coefficients of the equations of the hypersurfaces of degree n in P^γ.

The proof of Theorem 1 is in several stages.

Stage 1. For a general $V_{\gamma-1}^n$ the varieties $L_\gamma^{\alpha,\beta}$ and $\bar{L}_\gamma^{\alpha,\beta}$ are of zero dimension when $\alpha + \beta = n + 1$ and are empty when $\alpha + \beta < n + 1$.

Consider the incidence correspondence $z_1(z_2)$ between $M_\gamma^{\alpha,\beta}$ (or $\bar{M}_\gamma^{\alpha,\beta}$) and the space $P^{N(n,\gamma)}$ which, being a cycle $z_1 \subset M_\gamma^{\alpha,\beta} \times P^{N(n,\gamma)}$ (or $Z_2 \subset \bar{M}_\gamma^{\alpha,\beta} \times P^{N(n,\gamma)}$) set-theoretically consists of the pairs (l, V), where $l \in M_\gamma^{\alpha,\beta}$ (or $l \in \bar{M}_\gamma^{\alpha,\beta}$), $V \in P^{N(n,\gamma)}$ and l lies on the hypersurface V.

The family of hypersurfaces of degree n in P^γ passing through a straight line is a fiber of $Z_1(Z_2)$ over a point $M_\gamma^{\alpha,\beta}(\bar{M}_\gamma^{\alpha,\beta})$. This family is a linear space of dimension $\binom{n+\gamma}{\gamma} - (n+1)$.

The variety Z_1 is irreducible since it can be stratified over the irreducible variety $M_\gamma^{\alpha,\beta}$ on projective spaces. The variety Z_2 consists of two components corresponding to the components of the variety $\bar{M}_\gamma^{\alpha,\beta}$.

The image of the correspondence $z_1(z_2)$ is the whole space of hypersurfaces. Indeed, it easily follows from [1] that when $\alpha + \beta = n + 1$ we have $s_1(V_{\gamma-1}^n \cdot \Omega_{1,\alpha+\beta}) > 0$, and since $\Omega_{1,\alpha+\beta}$ is in the decomposition of $M_\gamma^{\alpha,\beta}$ with coefficient 1, then $(s_1(V_{\gamma-1}^n) \cdot M_\gamma^{\alpha,\beta}) > 0$.

$L_\gamma^{\alpha,\beta}$ ($\bar{L}_\gamma^{\alpha,\beta}$) is a fiber of z_1 (z_2) over a general point of the image under a projection on $P^{N(n,\gamma)}$. The principle for calculating the constants of z_1 (z_2) yields

$$\alpha+\beta+\binom{n+\gamma}{\gamma}-(n+1)=\binom{n+\gamma}{\gamma} + \dim L_\gamma^{\alpha,\beta} (\cdot \text{or} \cdot \bar{L}_\gamma^{\alpha,\beta}),$$

from which, when $\alpha + \beta = n + 1$, we obtain $\dim L_\gamma^{\alpha,\beta} = \dim \bar{L}_\gamma^{\alpha,\beta} = 0$. If, for $\alpha + \beta < n + 1$, the image of Z_1 (Z_2) were the whole space $P^{N(n,\gamma)}$, the same equation would hold. However, when $\alpha + \beta < n + 1$ it is impossible.

Stage 2. If $\alpha + \beta = n + 1$, then

$$N_\gamma^{\alpha,\beta} = N_{\gamma+1}^{\alpha,\beta} - N_{\gamma+1}^{n+1-\gamma,\gamma}, \tag{2}$$

$$N_\gamma^{\alpha,\beta} = N_{n+1}^{\alpha,\beta} - N_{n+1}^{n+1-\gamma,\gamma}. \tag{3}$$

Equation (3) follows from (2) by reverse induction on γ. We shall prove (2).

Let P^α and P^β lie in P^γ so that they generate the whole of P^γ. Consider the straight line C in $P^{\gamma+1}$, not lying in P^γ and not intersecting $P^{\beta-1}$. Each point $t \in C$ generates not only $P^{\beta-1}$, but also P_t^β. $P^{\beta+1}$ denotes the space generated by $P^{\beta-1}$ and C.

Let U be a correspondence between C and $L_{\gamma+1}^{\alpha,\beta+1}$, constructed for a general V_γ^n and for the P^α and $P^{\beta+1}$ chosen above, defined set-theoretically as the set of pairs (l, t) such that $l \in L_\gamma^{\alpha,\beta}$, constructed for P^α and P_t^β.

Since, by stage 1, $L_{\gamma+1}^{\alpha,\beta+1}$ is a fiber of an irreducible correspondence over $P^{N(n,\gamma+1)}$, there is an open set in $P^{N(n,\gamma+1)}$, for which $L_{\gamma+1}^{\alpha,\beta+1}$ is of the same size.

A fiber of the projection $U \to L_{\gamma+1}^{\alpha,\beta+1}$ is either a straight line (for $l \in L_{\gamma+1}^{\alpha,\beta+1}$) or lying in $P^{\beta+1}$ or intersecting $P^\alpha \cap P^{\beta-1}$) or a point.

Since stage 1 asserts that in C there is an open set C', and that a fiber of the projection $U \to C$ over points of C' is of zero dimension, there are no general points of the cycle U in the fibers of this morphism over C'. Hence the projection $U \to C$ over C' is a plane morphism.

$L_{\gamma+1}^{\alpha,\beta}$, is a fiber of U over the point of intersection of C and P^γ and, by stage 1, this point belongs to C', while $\bar{L}_{\gamma+1}^{\alpha,\beta}$ is a fiber over the remaining points. Because the morphism is plane it follows that

$$\chi(L_{\gamma+1}^{\alpha,\beta}) = \chi(\bar{L}_{\gamma+1}^{\alpha,\beta}).$$

$\chi(L_{\gamma+1}^{\alpha,\beta})$ is obviously $N_{\gamma+1}^{\alpha,\beta}$. The variety $\bar{L}_{\gamma+1}^{\alpha,\beta}$ consists of straight lines in P^γ which intersect P^α and P^β and of straight lines of P^γ intersecting $P^\alpha \cap P^\beta$. The number of straight lines of the first type is $N_\gamma^{\alpha,\beta}$ and of the second $N_{\gamma+1}^{n+1-\gamma,\gamma}$.

By stage 1, there are no straight lines on a general $V_{\gamma-1}^n$ intersecting $P^\alpha \cap P^\beta$ for $\alpha + \beta = n + 1$. Hence

$$N_{\gamma+1}^{\alpha,\beta} = N_\gamma^{\alpha,\beta} + N_{\gamma+1}^{n+1-\gamma,\gamma},$$

as was asserted.

Stage 3. Calculation of the numbers

Each straight line in P^{n+1}, intersecting P^α and P^β and lying on a hypersurface of degree n is defined by the two points $(x_0, \ldots, x_\alpha)$ and $(y_0, \ldots, y_\beta)$ in which it intersects P^α and P^β (we assume that the point of intersection of P^α and P^β does not lie on that hypersurface).

If the equation of P^α has the form $z_{\alpha+1} = \ldots = z_{\alpha+\beta} = 0$, and that of P^β: $z_0 = \ldots z_{\alpha+1} = 0$, the equations of the straight line have the form

$$z_0 = x_0 u, \ldots, z_\alpha = x_\alpha u + y_0 v, \ z_{\alpha+1} = y_1 v, \ldots, z_{\alpha+\beta} = y_\beta v. \tag{4}$$

The conditions that the straight line should belong to the hypersurface are obtained by equating to zero the coefficients of $u^\xi v^\eta$ $(\xi + \eta = n + 1)$ in the polynomial obtained by substituting the variables (4) in the equation of the hypersurface V_n^n. Thus, $L_{n+1}^{\alpha,\beta}$ lies on $P^\alpha \times P^\beta$ and is the intersection of divisors of bidegrees $(n, 0), (n - 1, 1), \ldots, (0, n)$.

The index of their intersection is $N_\gamma^{\alpha,\beta}$ and it is obviously equal to the coefficient of $x^\alpha y^\beta$ in the polynomial $nx[(n-1)x+y]\dots[x+(n-1)y]ny$. This coefficient is equal to the coefficient of $z^{\alpha-1}$ in the polynomial

$$n(1+(n-1)z)\dots nz, \tag{5}$$

which is the $(\alpha-1)$th elementary symmetric polynomial of the roots of (5) which are $-1/(n-1), -2/(n-2), \dots, -(n-1)/1$. Theorem 1 follows from this and stage 2.

Now the variety of straight lines in P^r lying on V^n_{r+1} is $L_r^{r-1,r-1}$. Application of Theorem 1 yields Theorem 2 since $2r-n-3=0$.

We now derive Theorem 3 from Theorem 1. We know that the basis of the cycles of a Grassman variety with respect to the modulus of numerical equivalence is comprised of Schubert varieties $\Omega_{a_0 a_1}$. Hence, for any r and n,

$$s_1(V^n_{r-1}) = L_r^{r-1,r-1} \sim \sum_{\max(0,\, r-n-2)\leqslant i< r-\frac{n}{2}-1} \alpha_i \Omega_{i,\,2r-n-2-i} \tag{6}$$

Moreover, $(\Omega_{i,\,2r-n-2-i};\ \Omega_{n+2+i_0-r,\,r-i_0})$ is unity for $i=i_0$ and zero otherwise [2]. Hence it follows from (6) that

$$\alpha_i = (L_r^{r-1,r-1}, \Omega_{n+2+i-r,\,r-i}) = N_{r-i}^{n+2+i-r,\,r-i-1}.$$

The variety of straight lines on the intersection of hypersurfaces is the intersection of the varieties of straight lines on each hypersurface. Hence

$$s_1(V_{r-s}^{n_1,\dots,n_s}) = (s_1(V^{n_1}_{r-1})\dots s_1(V^{n_s}_{r-1})).$$

Calculation of this index using (6) yields Theorem 3.

§2. Straight Lines on a Cubic Three-Dimensional Hypersurface. Let $h_X(n)$ denote in what follows the Hilbert polynomial of the variety X.

Arguing as in the proof of stage 2, we see that $h_{L_6^{3,3}} = h_{\bar L_6^{3,3}}$.

The variety $\bar L_6^{3,3}$ is the union of two varieties: $L_5^{3,3}$ and $L_6^{1,5}$, which intersect in $L_5^{1,4}$. Hence

$$h_{L_6^{3,3}} = h_{L_5^{3,3}} + h_{L_6^{1,5}} - h_{L_5^{1,4}}. \tag{7}$$

Moreover, $h_{L_5^{3,3}} = h_{\bar L_5^{3,3}}$. The variety $\bar L_5^{3,3}$ is the union of the varieties $L_4^{3,3}$ and $L_5^{2,4}$, which intersect in $L_4^{2,3}$. Hence

$$h_{L_5^{3,3}} = h_{L_5^{2,4}} + h_{L_4^{3,3}} - h_{L_4^{2,3}}. \tag{8}$$

Similarly, $h_{L_6^{2,4}} = h_{\bar L_6^{2,4}}$. The variety $\bar L_6^{2,4}$ splits into $L_5^{2,4}$ and $L_6^{1,5}$, which intersect in $L_5^{1,4}$. Hence

$$h_{L_6^{2,4}} = h_{L_5^{2,4}} + h_{L_6^{1,5}} - h_{L_5^{1,4}}. \tag{9}$$

Combining (7)-(9), we obtain

$$h_{L_4^{3,3}} = h_{L_6^{3,3}} - h_{L_6^{2,4}} + h_{L_4^{2,3}}. \tag{10}$$

Now we calculate each of the polynomials on the right side.

For any cycle Z in $P^3\times P^3$, let i_Z denote its embedding in $P^3\times P^3$, let P_1 and P_2 denote the projections of $P^3\times P^3$ on each of the factors.

The variety $L_6^{3,3}$ is embedded in $P^3\times P^3$ and is the intersection of divisors of bidegrees (3,0), (2, 1), (1, 2), (0, 3).

Let D_1 be the divisor of bidegree (3, 0). Then we have

$$0\to p_1^*(O_{P^3}(-3))\to O_{P^3\times P^3}\to O_{D_1}\to 0.$$

Forming the tensor product of this sequence with $p_1^*(O(n))\otimes p_2^*(O(m))$, we obtain

In particular, $p_a(L_4^{3,3}) = 5$, $\deg L_4^{3,3} = 45$, which is Fano's classical result [3].

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