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On the Fundamental Group of the Space of Cubic Surfaces

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The space of complex cubic surfaces is isomorphic to projective space V_{19} of dimension 19, by the correspondence which relates to each surface the set of coefficients of the defining equation. We denote by Δ the hypersurface in V_{19} which consists of points corresponding to the singular surfaces. It was proved by J.A. Todd that there exists a homomorphism t of the fundamental group $\pi_1(V_{19}-\Delta, p_0)$ onto the Weyl group $W(E_6)$ of system of roots of type E_6. This homomorphism is actually just the one which relates to each loop the permutation of 27 straight lines induced by moving of the surface corresponding to p_0 around the loop.

The purpose of this paper is to describe the "upper boundary" for $\pi_1(V_{19}-\Delta, p_0)$. Namely, we prove the following

Theorem. *There exists a homomorphism h of the Brieskorn braid group $B(E_6)$ corresponding to the system of roots of type E_6 onto $\pi_1(V_{19}-\Delta, p_0)$ such that the following diagram is comutative*

$$\begin{array}{ccc} B(E_6) & & \\ {\scriptstyle h}\downarrow & \searrow^{p} & \\ \pi_1(V_{19}-\Delta, p_0) & \xrightarrow{t} & W(E_6) \end{array}$$

where p is the canonical projection of the braid group onto the Weyl group.

The main point of our proof is the observation that the intersection form in 2-dimensional homology of a cubic surface restricted to the orthogonal complement to the class of a plane section can be identified with the intersection form of the smoothing of a certain E_6-singularity (see [5] where this approach was used for studying decompositions of high-dimensional hypersurfaces into connected sums). It gives the explanation of the connection between the group $W(E_6)$ and cubic surfaces from the point of view of singularities (compare [6]).

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1) First note that the intersection form of the smoothing of the singularity (see [1] for definitions)

$$Z_1^3+Z_2^2+Z_2Z_3^2=0 \tag{1}$$

is one associated to the Dynkin diagram of type E_6. Indeed, $Z_1^3+Z_2^2+Z_2Z_3^2 = Z_1^3+(Z_2+\frac{1}{2}Z_3^2)^2-\frac{1}{4}Z_3^4$ and therefore an invertible change of variables shows that this singularity is equivalent to the singularity $u^3+v^2+w^4=0$ for which the intersection form computed in [3].

2) Let us consider the semiuniversal deformation of the singularity (1). Recall that for an arbitrary isolated singularity $f(Z_1 \dots Z_n)$ it can be described as follows. Let $1, g_1, \dots, g_k$ be a basis of the artinian ring $R=\mathbb{C}\{Z_1 \dots Z_n\}/(f, \partial f/\partial Z_1 \dots \partial f/\partial Z_n)$ and $F(Z,t)=f(Z)+\sum_{i=1}^{k} g_i(Z)\,t_i$. Then the germ of the map $\mathbb{C}^n \times \mathbb{C}^k \to \mathbb{C}^1 \times \mathbb{C}^k$ defined by the formula $(Z,t) \rightsquigarrow (F(Z,t),t)$ is the germ of a semiuniversal deformation of the singularity f.

Easy computation shows that for the singularity $Z_1^3+Z_2^2+Z_2Z_3^2=0$ the monomials $1, Z_1, Z_2, Z_3, Z_1Z_2, Z_1Z_3$ form a basis of the ring R, and therefore the germ of the semiuniversal deformation for the singularity (1) is $(Z_1, Z_2, Z_3, t_1, t_2, t_3, t_4, t_5) \to (Z_1^3+Z_2^2+Z_2Z_3^2+t_1Z_1+t_2Z_2+t_3Z_3+t_4Z_1Z_2+t_5Z_1Z_3, t_1, t_2, t_3, t_4, t_5)$.

3) It was pointed out by E. Brieskorn that for simple singularities the subset of the base of a semiuniversal deformation over which the deformation is non-singular fibration (i.e. complement to the bifurcation variety) can be identified with the set of regular semisimple conjugacy classes in an appropriate simple Lie group. According to [2] the fundamental group of the set of regular semisimple classes is the Brieskorn braid group, i.e. it admits the set of generators which correspond one-to-one to a system of simple roots with the relations

$$\underbrace{\alpha\beta\alpha\dots}_{m_{\alpha\beta}} = \underbrace{\beta\alpha\beta}_{m_{\alpha\beta}}$$

where $m_{\alpha\beta}$ is the Coxeter matrix.

4) Note that the set $\mathbb{C}^6-\Delta$ of the points $(t_1 \dots t_6)$ for which the affine surface

$$\begin{aligned} &F(Z_1, Z_2, Z_3, t_1 \dots t_5) \\ &\quad = Z_1^3+Z_2^2+Z_2Z_3^2+t_1Z_1+t_2Z_2+t_3Z_3+t_4Z_1Z_2+t_5Z_1Z_3=t_6 \end{aligned} \tag{2}$$

is non-singular is a deformation retract of the complement to the bifurcation variety of the singularity (1) because the polynomial $F(Z_1, Z_2, Z_3, t_1 \dots t_5)$ is weighted homogeneous (of the weight (4, 6, 3, 8, 6, 9, 2, 5) and degree 12), i.e. admits $\mathbb{C}^*$-action. Therefore $\pi_1(\mathbb{C}^6-\Delta, p_0)$ is isomorphic to Brieskorn braid group of type E_6.

Let V_6 denote the 6-dimensional linear system of cubic surfaces

$$\begin{aligned} &A_0(Z_1^3+Z_2^2+Z_2Z_3^2)+A_1Z_0^3+A_2Z_0^2Z_1+A_3Z_0^2Z_2+A_4Z_0^2Z_3 \\ &\quad +A_5Z_0Z_1Z_2+A_6Z_0Z_1Z_3=0 \end{aligned} \tag{3}$$

which are the projective closure of the surfaces (2).

We claim that $\mathbb{C}^6-\Delta=V_6-\Delta$. Indeed, direct computation shows that $V_6(A_0\dots A_6)$ is singular only if $A_0=0$ and if the singularity belongs to the set $Z_0\neq 0$ in $\mathbb{CP}^3$. In particular, $\pi_1(V_6-\Delta)\simeq B(E_6)$.

Remark. V_6 is not in general position with respect to Δ and therefore we cannot apply Zariski's theorem [4].

5) Now let us consider the following families V_7, V_{10} and V_{11} of cubic surfaces in $\mathbb{CP}^3$. Let V_7 be the family

$$\begin{aligned} &A_0(Z_1^3+Z_2Z_3^2)+BZ_2^2Z_0+A_1Z_0^3+A_2Z_2Z_0^2+A_3Z_3Z_0^2 \\ &\quad +A_4Z_1Z_0^2+A_5Z_0Z_1Z_2+A_6Z_0Z_1Z_3=0. \end{aligned} \tag{4}$$

Let V_{10} be the family

$$A_0(Z_1^3+Z_2Z_3^2)+Z_0P_2(Z_0,Z_1,Z_2,Z_3)=0 \tag{5}$$

where P_2 is a general homogeneous polynomial of degree 2.

Let V_{11} be the family

$$A_0Z_1^3+B_0Z_2Z_3^2+Z_0P_2(Z_0,Z_1,Z_2,Z_3)=0 \tag{6}$$

where $P_2(Z_0,Z_1,Z_2,Z_3)$ is as above.

We claim that

$$\pi_1(V_6-\Delta)=\pi_1(V_7-\Delta)=\pi_1(V_{10}-\Delta)=\pi_1(V_{11}-\Delta).$$

Indeed, V_6 is the element of the pencil $V_6^{(u,v)}$ of hyperplanes in V_7 defined by the equation $A_0u=Bv$, which corresponds to $u=v$. Evidently $V_6^{(u,v)}-\Delta$ is isomorphic to $V_6-\Delta$ for each (u,v) where $u\neq 0$ and $v\neq 0$. (This isomorphism is defined by the change of variables $Z_0'=\frac{u}{v}Z_0$.) Therefore, V_6 is a general hyperplane with respect to Δ, and by Zariski's theorem ([4])

$$\pi_1(V_6-\Delta)\simeq\pi_1(V_7-\Delta).$$

Similar arguments show that

$$\pi_1(V_{11}-\Delta)\simeq\pi_1(V_{10}-\Delta).$$

Now the group $\mathbb{C}^3$ acts on V_{10} in the following way

$$\begin{aligned} (m,n,k)\colon (&Z_1\to Z_1+mZ_0,\ Z_2\to Z_2+nZ_0, \\ &Z_3\to Z_3+nZ_0,\ Z_0\to Z_0). \end{aligned}$$

Direct computation shows that $V_{10}-\Delta/\mathbb{C}^3$ is isomorphic to $V_7-\Delta$ (i.e. each non-singular surface of form (5) can in a unique way be reduced to the "normal form" (4)). Therefore, $\pi_1(V_{10}-\Delta)\simeq\pi_1(V_7-\Delta)$ and our claim is proved.

6) Recall that to each non-singular cubic surface one can associate a parabolic curve ([7]), i.e. the set of the planes intersection of which with the cubic surface V is a cubic curve with at least a cusp. It is known that the parabolic curve of a non-singular cubic surface is in general non-singular and admits ordinary singularities exactly for surfaces with Eckardt points ([7], § 70).

Now let us consider the space T of the planes in $\mathbb{C}\mathbb{P}^3$ which are endowed by the following configuration.

(a) two non-coincident straight lines l_1 and l_2.

(b) the points a_1 on l_1 and a_2 on l_2 such that $a_1 \neq a_2 \neq l_1 \cap l_2$.

It is easy to check that the space of planes with such configuration is simply connected and that $\pi_2(T)=\mathbb{Z}^3$. Indeed, $PGL(3,\mathbb{C})$ acts transitively on T and the stabilizer of each point is the image of the subgroup of the matrices

$$\begin{pmatrix} a_{11} & & & 0 \\ 0 & a_{22} & & \\ & & a_{33} & \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

in $PGL(3,\mathbb{C})$ under the canonical projection $GL(4,\mathbb{C})\to PGL(3,\mathbb{C})$. Our assertion follows from the well known computation of the homotopy groups of $PGL(3,\mathbb{C})$.

7) Let us consider the subvariety S of $T\times(V_{19}-\Delta)$ which consists of the pairs (t, V) where $t\in T$ and V is the non-singular cubic surface such that

(a) $V\cap t$ is a cubic curve with a cusp at the point $a_1\in t$,

(b) l_1 is the double tangent at this cusp,

(c) a_2 is the unique inflexion point of the cubic curve $V\cap t$ and l_2 is the tangent at the point a_2.

Let us consider the projection $S\to T$. All the fibres are isomorphic, because $PGL(3,\mathbb{C})$ acts transitively on T. The fibre over the plane $Z_0=0$ endowed with the configuration

$$l_1=\{Z_0=0,\ Z_3=0\}, \qquad l_2=\{Z_0=0,\ Z_1=0\},$$
$$a_1=\{Z_3=0,\ Z_1=0,\ Z_0=0\}, \qquad a_2=\{Z_1=0,\ Z_2=0,\ Z_0=0\}$$

is isomorphic to $V_{11}-\Delta$. Hence from the exact homotopy sequence of the fibration it follows that $\pi_1(S)$ is a factor of $\pi_1(V_{11}-\Delta)$, i.e., $B(E_6)$.

8) Now let us consider the projection $S\to V_{19}-\Delta$. The fibre over V is actually a parabolic curve of V. Therefore, over a dense open set of $V_{19}-\Delta$ this mapping is a locally trivial bundle.

Hence the induced map $\pi_1(S)\to\pi_1(V_{19}-\Delta)$ is onto.

9) The composition of the homomorphism $h: B(E_6) \to \pi_1(V_{19} - \Delta, p_0)$ and the Todd homomorphism t gives the map $B(E_6) \to W(E_6)$. On the other hand, there is the canonical projection $p: B(E_6) \to W(E_6)$ ([9], 1.2), which in terms of the presentation 3) can be described as the map which takes the square of each generator to the identity.

Now we are going to show that $t \cdot h$ actually coincide with p.

According to [2] the generators of $B(E_6)$ are represented by the loop which surrounds a non-singular point of the discriminant variety of the space of regular orbits. The non-singular points of the discriminant in the space of cubic surfaces corresponds to the cubic surfaces with a single ordinary singularity ([7]). Going around such a point induces a permutation of some Schlafli double six on the cubic surface which corresponds to p_0 ([8]). Therefore, the square of each generator of $B(E_6)$ induces the identity permutation of 27 lines.

This concludes the proof of the theorem.

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References

1. Brieskorn, E.: Singular elements of semi-simple algebraic groups. Actes du Congrès International des Mathématiciens (Nice 1970) vol. 2, pp. 279–284. Paris: Gauthier-Villars 1971
2. Brieskorn, E.: Die Fundamentalgruppe des Raumes der regularen Orbits einer endlichen komplexen Spiegelungsgruppe. Invent. Math. **12**, 57–61 (1971)
3. Gabrielov, A.: Intersection matrices of certain singularities. Functional Anal. Appl. **7**, 182–193 (1973)
4. Lê Dung Trang: Une application du theoreme de Zariski. C.R. Acad. Sci. Paris Sér. A **273**, 57 (1972)
5. Libgober, A.: On the topology of some even-dimensional algebraic hypersurfaces. Preprint
6. Manin, Y.I.: Cubic forms. Amsterdam-London: North-Holland 1974
7. Segre, B.: The Non-singular cubic surfaces. Oxford: Clarendon Press 1942
8. Todd, J.A.: On the Topology of Certain Algebraic Threefold Loci. Proc. Edinburgh Math. Soc. ser. **2**, 4, 175–184 (1935)
9. Brieskorn, E., Saito, K.: Artin-Gruppen und Coxeter-Gruppen. Invent. Math. **17**, 245–271 (1972)

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