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ON THE TOPOLOGY OF SOME EVEN-DIMENSIONAL ALGEBRAIC HYPERSURFACES

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§1. INTRODUCTION

THE TOPOLOGICAL structure of odd-dimensional hypersurfaces was investigated recently by several authors[5], [9], [10]. These results are the generalization of the classical handle-decomposition of Riemann surfaces and can be summarized as follows.

THEOREM. *Let V_n^d be a non-singular algebraic hypersurface of an odd dimension n and of degree d in an $(n+1)$-dimensional complex projective space. Let T denote the* PL-*manifold which is obtained by gluing a cone to the spherical boundary of the plumbing of two copies of the tangent bundles of the sphere S^n. Then there are two cases*

(i) *If* $d \not\equiv \pm 3 \pmod 8$ *or* $n = 1, 3, 7$, *then*

$$V_n^d = k(S^n \times S^n) \# M_n^d$$

(ii) *If* $d \equiv \pm 3 \pmod 8$ *and* $n \neq 1, 3, 7$, *then*

$$V_n^d = (k-1)(S^n \times S^n) \# T \# M_n^d$$

where $k = (1/2) rk H_n(V_n^d, \mathbf{Z})$ *and* M_n^d *is a manifold (generally piecewise-linear in case* (ii)) *which has the same homology as the projective space* $\mathbf{CP}^n$.

Note also that M_n^d can either be realized as a rational singular projective hypersurface[5] or as a gluing of two copies of some D^{n+1} disk bundle over $\mathbf{CP}^{n-1/2}$ (see Remark 5 below).

The topology of even-dimensional hypersurfaces is considered in [4], where it is proved that it is possible to split off the maximal number of the handles $S^n \times S^n$. The manifold which remains after removing these handles is in general decomposable into a connected sum of more simple manifolds.

In this paper we describe the decomposition of some even-dimensional algebraic hypersurfaces into the connected sum of indecomposable almost differentiable manifolds (PL-manifolds with the differentiable structure on the complement of a point).

Let $F_{2n}^{\circ}(E_8)$ be the $2n$-dimensional manifold which is the plumbing of the tangent bundles of the sphere S^n according to the graph E_8[3]. The boundary of $F_{2n}^{\circ}(E_8)$ is homeomorphic to a sphere and we denote by $F_{2n}(E_8)$ the closed PL-manifold which is obtained by adding the cone over the sphere S^{2n-1} to the boundary of $F_{2n}(E_8)$. Let V_n^d be hypersurface of dimension n and degree d in $\mathbf{CP}^{n+1}$. Our main result is the following:

THEOREM. *Suppose that* $n > 2$, $n \equiv 2 \pmod 4$ *and d is even. Then* $V_n^d \cong a(S^n \times S^n) \# bF_{2n}(E_8) \# M_n^d$ *where* $\cong$ *stands for a* PL-*homeomorphism*, $a =$

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$(1/2)(rkH_n(V_n^d)-|\text{sign } V_n^d|-2)$ and $b=(1/8)|\text{sign } V_n^d|$. The manifold M_n^d can be obtained by gluing two copies of a certain D^n-bundle over $\mathbf{CP}^{n/2}$ by means of a PL-homeomorphism of the boundary.

In conclusion we formulate some results concerning the decomposition of other classes of even-dimensional hypersurfaces; their proof is based on the same ideas used in the proof of the main theorem.

§2. PROOF OF THE MAIN THEOREM

Recall that all non-singular hypersurfaces of a given degree and dimension are diffeomorphic. So it is enough to describe the decomposition for some special model of a hypersurface V_n^d. Let us consider the hypersurface $V_n^d(c)$ which is the projective completion of the affine hypersurface defined by the equation

$$P_n(Z_1 \ldots Z_{n+1}) = Z_1^d + Z_2^{d-1} + Z_2 Z_3^{d-1} + \cdots + Z_{n-1}Z_n^{d-1} + Z_n Z_{n+1}^{d-1} = c.$$

The properties of this hypersurface are considered in [5]. For c sufficiently close to zero we denote by F_n^d the intersection of $V_n^d(c)$ with the ball B_ϵ of a small radius ϵ centered at the point with coordinates $Z_1 = Z_2 = \cdots = Z_{n+1} = 0$. Let

$$G_n^d = \overline{V_n^d(c) - F_n^d}.$$

PROPOSITION [5] (i). *The hypersurface $V_n^d(c)$ is a non-singular projective variety for $c \neq 0$ and has a single isolated singularity for $c=0$ at the point $Z_1 = \cdots Z_{n+1} = 0$.*

(ii). *F_n^d is an $(n-1)$-connected parallellizable $2n$-manifold with boundary and its n-dimensional Betti number b_n is given by*

$$b_n = \frac{1}{d}[(d-1)^{n+2} + (-1)^n(d-1)].$$

(iii). *The characteristic polynomial $\Delta_n(t)$ of the monodromy of the isolated singularity of $V_n^d(0)$ can be computed by the recursive equation*

$$\Delta_{n+1}(t) = \Delta_n^{-1}(t)\frac{t^{d(d-1)^{n+1}}-1}{t^{(d-1)^{n+1}}-1}.$$

Sketch of the proof. (i) can be verified by direct computation. (ii) is a consequence of the fact that the polynomial $P_n(Z_1 \ldots Z_{n+1})$ is weighted homogeneous of the weight

$$\left(d, d-1, \ldots, \frac{d(d-1)^i}{(d-1)^i + (-1)^{i-1}}, \ldots, \frac{d(d-1)^n}{(d-1)^n + (-1)^{n-1}}\right).$$

(iii) can be checked by using the Milnor–Orlik[8] algorithm for the computation of a characteristic polynomial of the weighted homogeneous singularities[5].

From now on we consider the case n is even and $n > 2$. We denote by $H_n(V_n^d, \mathbf{Z})_0$ the group of vanishing cycles, i.e. $\text{Ker}(H_n(V_n^d, \mathbf{Z}) \to H_n(\mathbf{CP}^{n+1}, \mathbf{Z}))$. This group can be also described as the image of the Hurewicz homomorphism $\pi_n(V_n^d) \to H_n(V_n^d)$ or as the orthogonal complement to the homology class h of the intersection of V_n^d and $\mathbf{CP}^{n/2+1}$[4]. Let us denote by l the projective space defined by $Z_1 = \sqrt[d]{c}Z_0$, $Z_2 = Z_4 =$

$\cdots = Z_n = 0$ belonging to $V_n^d(c)$. Note that the intersection of l with the small ball B_ϵ centered at the point with coordinates $Z_1 = Z_2 = \cdots = Z_n = Z_{n+1} = 0$ is a disk. Denote this disk by l_1 and $\overline{l - l_1}$ denote by l_2. By abuse of notation we let l_1 (resp. l_2) also denote the relative homology class in $H_n(F_n^d, \partial F_n^d)$ (resp. in $H_n(G_n^d, \partial G_n^d)$) defined by l_1 (resp. l_2).

LEMMA 1. (a) *The homomorphism*

$$i_F: H_n(F_n^d, \mathbf{Z}) \to H_n(V_n^d, \mathbf{Z})$$

is an imbedding and its image is the subgroup of vanishing cycles, $H_n(V_n^d, \mathbf{Z})_0$.

(b) *The group* $H_n(G_n^d, \mathbf{Z})$ *is generated by* h *and the group* $H_n(G_n^d, \partial G_n^d, \mathbf{Z})$ *by* l_2. *In* $H_n(G_n^d, \partial G_{n-1}^d, \mathbf{Z})$ *holds the relation* $dl_2 = h$.

(c) ∂F_n^d *is a* $\mathbf{Q}$*-homology sphere with* $H_{n-1}(\partial F_n^d, \mathbf{Z}) = \mathbf{Z}/d$.

Proof. (a) From (iii) of the proposition we conclude that $\Delta_n(1) = d$ for n even. Hence $H_{n-1}(\partial F_n^d)$ is finite of order d [6]. By Poincare duality $H_n(\partial F_n^d) = 0$. We now consider the following Mayer–Vietoris sequence

$$0 \to H_n(F_n^d) \oplus H_n(G_n^d) \to H_n(V_n^d) \to H_{n-1}(\partial F_n^d) \to 0. \tag{1}$$

It follows that i_F is an imbedding. The image of i_F belongs to the subgroup $H_n(V_n^d, \mathbf{Z})_0$ because the Hurewicz homomorphism is onto for the $(n-1)$-connected $2n$-manifolds. The rank of $H_n(V_n^d, \mathbf{Z})$ is equal to $(1/d)[(d-1)^{n+2} + (d-1)] + 1$ [4], [9]. Hence

$$rkH_n(V_n^d, \mathbf{Z})_0 = \frac{1}{d}[(d-1)^{n+2} + (d-1)] = rkH_n(F_n^d, \mathbf{Z})$$

by (ii) of the proposition. It follows that $i_F(H_n(F_n^d))$ has finite index in $H_n(V_n^d, \mathbf{Z})_0$.

From the diagram

$$\begin{array}{ccc} H_n(F_n^d) & \xrightarrow{i_F} & H_n(V_n^d, \mathbf{Z})_0 \\ \downarrow & & \downarrow \\ H_n(F_n^d, \mathbf{Z})^* & \longleftarrow & H_n(V_n^d, \mathbf{Z})_0^* \end{array}$$

we obtain that this index equals 1 because

$$[H_n(F_n^d) : H_n(F_n^d)^*] = [H_n(V_n^d)_0 : H_n(V_n^d)_0^*] = d$$

(where for an abelian group E we denote by E^* the dual group Hom $(E, \mathbf{Z})$).

(b) It follows from the sequence (1) that $H_n(G_n^d, \mathbf{Z})$ is a free cyclic group. Because the number of elements in $H_{n-1}(\partial G_n^d) = H_{n-1}(\partial F_n^d)$ is d we obtain that the square of the generator of $H_n(G_n^d, \mathbf{Z})$ is equal to $\pm d$, i.e. h is a generator of $H_n(G_n^d, \mathbf{Z})$. The other assertions of (b) follows from this.

(c) The manifold ∂F_n^d is $(n-2)$-connected [6] and because $H_{n-1}(\partial F_n^d, \mathbf{Z})$ is a torsion group (see (a)) we obtain the first part of the assertion. The second one follows from (b).

LEMMA 2. *If* d *is even and* $n \equiv 2 \pmod 4$, *then there exists an element* $v \in H_n(F_n^d, \partial F_n^d, \mathbf{Z})$ *such that* $v^2 = -1/d$.

Proof. For any $2n$-dimensional manifold X we denote by $S(H_n(X))$ the intersection form on $H_n(X, \mathbf{Q})$. Since ∂F_n^d is a rational homology sphere we have ([3] p. 48).

$$(-S(H_n(F_n^d)) \oplus (S(H_n(G_n^d)) = S(H_n(V_n^d)). \tag{2}$$

Moreover it is computed in [4] that $l^2 = (1/d)[1-(1-d)^{(n/2)+1}]$. Since $l_2 = (1/d)h$ and $h^2 = d$, we have $l_2^2 = 1/d$. Hence $l_1^2 = (1/d)[1-(1-d)^{(n/2)-1}] - (1/d)$. According to [4], for $d > 2$ there exist elements e_1 and e_2 in $H_n(V_n^d, \mathbf{Z})_0$ and hence in $H_n(F_n^d, \mathbf{Z})$ such that $e_1^2 = e_2^2 = 0$ and $e_1e_2 = 1$. Let $a = (l_1, e_1)$ and $b = (l_1, e_2)$. Then the element

$$v = l_1 + \left(b(a-1) - \frac{1}{2d}[1-(1-d)^{(n/2)+1}]\right)e_1 + (1-a)e_2$$

has square $-1/d$. Note that under the assumption of the lemma the coefficient of e_1 is an integer.

LEMMA 3. *Let E be a free abelian group endowed with a non-singular integer symmetric bilinear form β. Let $E^*/E = \mathbf{Z}/d$. Then β induces on E^* a bilinear form with values in $\mathbf{Z}_{(d)}$-subgroup of Q, which consists of the fractions with the divisors of d as denominator. If there also exists an element $v \in E^*$ such that $v^2 = \pm 1/d$ then there exists an orthogonal decomposition $E^* = A \oplus \{v\}$ where A is an inner product space and $\{v\}$ is a subgroup generated by v.*

Proof. This is a special case of Theorem 3.2 from [7].

Now we are ready to conclude the proof of the main theorem. Note that the class v which was built during the proof of Lemma 2 can be represented by an immersed disk whose boundary is the sphere ∂l_1. According to *A*. Haefliger's theorem for manifolds with boundary ([1] Theorem 4.1) we may suppose that v is realizable by an embedded disk with the same boundary.

Let T denote the tubular neighbourhood of this disk. Let us consider the manifolds $\bar{M}_n^d = G_n^d \cup T$ and $\bar{N}_n^d = \overline{F_n^d - T}$. The abelian group $H_n(\bar{N}_n^d, \mathbf{Z})$ endowed with the intersection form is isomorphic to A from Lemma 3, which built for $E = H_n(F_n^d, \mathbf{Z})$. It follows that the intersection form on $H_n(\bar{N}_n^d, \mathbf{Z})$ is unimodular, even and indefinite. Because $\bar{N}_n^d$ is parallelizable ((ii) of proposition) it follows that $\bar{N}_n^d = a(S^n \times S^n) \natural\, bF^{-}(E_8)$ ($\natural$ denotes the boundary connected sum.)

Now we consider M_n^d which is $\bar{M}_n^d$ with added cone to the boundary. Let P_1 denote the union of l_2 with the disk in F_n^d which represents V. Obviously P_1 is diffeomorphic to the projective space $\mathbf{CP}^{n/2}$. It follows from (2) that the self-intersection index of P_1 is equal to zero. Let P_2 be obtained by a slight translation of P_1 in such a fashion that it does not intersect P_1. Let T_1 and T_2 be non-intersecting tubular neighbourhoods of P_1 and P_2 respectively. We prove that $\overline{M_n^d - T_1 - T_2}$ is an h-cobordism between ∂T_1 and ∂T_2. Indeed from the exact sequence of closed subspace

$$\to H_i(P_1) \to H_{i+n-1}(M - P_1) \to H_{i+n-1}(M) \to H_{i-1}(P_1) \to$$

it follows that $M_n^d - T_1$ has the same homotopy type as P_2. Using the exact sequence of the pair we obtain that

$$H_i(\overline{M_n^d - T_1 - T_2}, \partial T_2) = H_i(\overline{M_n^d - T_1}, T_2) = 0.$$

Hence the assertion of the theorem about M_n^d follows from the theorem about h-cobordism for PL-manifolds.

§3. SOME CONCLUDING REMARKS

Remark 1. The PL-homeomorphism which is mentioned in the main theorem is not in general smooth, because M_n^d does not generally admit a differential structure. For example for M_6^4 we obtain the following values of the Pontrjagin classes $p_1 = -8h^2$; $p_2 = 156h^4$ (h denotes the standard generator of $H^2(M_6^4, \mathbf{Z}) \simeq H^2(V_6^4, \mathbf{Z})$). Furthermore, the signature of M_6^4 is zero; thus if we assume that M_6^4 is a differentiable manifold we obtain [2]

$$0 = \text{sign}\,(M_6^4) = \frac{1}{945}(62p_3 - 13p_2p_1 + 2p_1^3)$$

and therefore p_3 cannot be an integer cohomology class.

Remark 2. The hypersurfaces V_n^d for which $n \equiv 2 \pmod 4$ and d is even are exactly the hypersurfaces whose intersection forms on $H_n(V_n^d, \mathbf{Z})$ have even type (see [4]).

Remark 3. If $n \equiv 0 \pmod 4$ or d is odd then a decomposition $V_n^d = N_n^d \# M_n^d$ where N_n^d is $(n-1)$-connected and $rkH_n(N_n^d) = rkH_n(V_n^d) - 2$ exists in the following cases. Either $d \equiv 0 \pmod 8$ (and then sign $M_n^d = 0$), or $d \equiv 2 \pmod 8$ and all prime divisors of d has form $4l+1$ (and then sign $M_n^d = 2$). Indeed, the existence of such a decomposition will imply that $H_n(F_n^d, \mathbf{Z})$ has a subgroup A of corank 1 on which the intersection form is unimodular. It would then follow that in $H_n(F_n^d, \partial F_n^d, \mathbf{Z})$ there exists an element v for which $v^2 = \pm 1/d$, but that is possible just in the cases listed above.

Note that non-existence of such a decomposition in the cases $d \not\equiv 0, 2 \pmod 8$ and $n \equiv 0 \pmod 4$ or d is odd and $n \equiv 2 \pmod 4$ follows from the fact that the intersection form on $H_n(V_n^d, \mathbf{Z})_0$ has even type together with the following lemma.

LEMMA 4. *If* $n \equiv 2 \pmod 4$ *and d is even then* sign $V_n^d \equiv 0 \pmod 8$, *otherwise* sign $V_n^d \equiv d \pmod 8$.

Proof. As above let l denote the homology class of $\mathbf{CP}^{n/2}$ in V_n^d and h denote the class of the intersection of V_n^d with $\mathbf{CP}^{n/2+1}$. Each element $c \in H_n(V_n^d, \mathbf{Z})$ can be represented as $c = c_0 + \kappa l$ where $c_0 \in H_n(V_n^d, \mathbf{Z})_0$ and $\kappa = (c, h)$. It follows then that $(h, c) \equiv (c, c) \bmod 2$ if $n \equiv 0 \pmod 4$ or $n \equiv 2 \pmod 4$ and d is odd. Hence from the ven der Blij lemma (see [7]) it follows that $d = h^2 \equiv \text{sign}\, V_n^d \pmod 8$.

If $n \equiv 2 \pmod 4$ and d is even then the intersection form on $H_n(V_n^d, \mathbf{Z})$ has even type and the congruence sign $V_n^d \equiv 0 \pmod 8$ follows from [7].

Remark 4. Nevertheless if $n = 4$ (resp. $n = 8$) we have the following decomposition

$$V_n^d \# P = N_n^d \# M_n^d$$

where P denotes the projective plane over the quaternions (resp. over the Caley numbers), N_n^d is an $(n-1)$-connected manifold and M_n^d is a gluing of two copies of a D^n-disk bundle over $\mathbf{CP}^{n/2}$, and $rkH_n(V_n^d \# P) = rkH_n(N_n^d) + 2$.

Remark 5. Let n be odd and $V_n^d = M_n^d \# N_n^d$ where $rkH_n(N_n^d) = rkH_n(V_n^d)$ and N_n^d is an $(n-1)$-connected manifold [5], [9], [10]. Then M_n^d also can be represented as a gluing of two copies of a D^{n+1}-bundle over $\mathbf{CP}^{n-1/2}$ by means of some homeomorphism of the boundary.

Indeed, let P_1 denote $\mathbf{CP}^{n-1/2}$ embedded in M_n^d such that the induced map $H_i(\mathbf{CP}^{n-1/2}, \mathbf{Z}) \to H_i(M_n^d, \mathbf{Z})$ is an isomorphism for $i \leq n-1$. Let P_2 be obtained by translation of P_1 in such a fashion that it does not intersect P_1. Let T_1 and T_2 be non-intersecting tubular neighbourhoods of P_1 and P_2 respectively. Then by a computation similar to the above of homology groups it can be shown that $\overline{M_n^d - T_1 - T_2}$ is an h-cobordism and therefore M_n^d is equivalent to a gluing of T_1 and T_2 by means of some homeomorphism of the boundaries.

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