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LEVINE'S FORMULA IN KNOT THEORY AND QUADRATIC RECIPOCITY LAW

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§ 1. Introduction

A k-knot is a k-dimensional submanifold of S^{k+2} which is homeomorphic to a sphere. Any knot K is bounded by a submanifold $F^{k+1} \subset S^{k+2}$ which is called the Seifert surface of K. One associates with K the Alexander polynomial $\Delta(t)$. Moreover if $k = 4n + 1$ then one may associate with F^{4n+2} the non-degenerate quadratic Z_2-form φ on $H_{2n+1}(F^{4n+2}, Z_2)$. Levine's formula asserts that the Arf invariant of this quadratic form is trivial if $\Delta(-1) \equiv \pm 1 \pmod 8$ and is non trivial if $\Delta(-1) \equiv \pm 3 \pmod 8$.

Levine's proof consists of two parts. The first one is topological and states that both the quadratic function φ on $H_{2n+1}(F^{4n+2}, Z_2)$ which is used for the computation of the Arf invariant, and the Alexander polynomial can be expressed in terms of the Seifert pairing L of F^{4n+2}, which is the bilinear form on $H_{2n+1}(F^{4n+2}, Z)$. Namely

$$\varphi(x \bmod 2) \equiv L(x, x) \bmod 2 \tag{1}$$

and

$$\Delta(t) = \det(L - tL^t) \tag{2}$$

i.e.

$$\Delta(-1) = \det(L + L^t). \tag{2a}$$

($L^t(x, y)$ is by definition $L(y, x)$).

The second part of Levine's proof is the remarkable observation that the Arf invariant of a quadratic function defined by (1) can be found in terms of the associated bilinear form $L + L^t$. He proved the following (cf. [6])

Levine's lemma. Let $L(x, y)$ denote a bilinear form on a free abelian group such that $d = \det(L+L')$ is odd. Let φ denote the quadratic function on $V \otimes Z_2$ defined by (1). Then

$$\operatorname{Arf}\varphi = \begin{cases} 1 & \text{if} \quad d \equiv \pm 1 \pmod 8 \\ -1 & \text{if} \quad d \equiv \pm 3 \pmod 8 \end{cases}$$

(We suppose that the range of Arf invariant is ± 1).

The purpose of this paper is to show that Levine's lemma is closely related to the Weil-Milgram reciprocity law ([4], [5]). In fact our main result is a generalization of Levine's lemma to arbitrary algebraic number fields.

Let F denote an algebraic number field and L be a bilinear form on a projective module P over the ring R of integers in F. Suppose that the determinant of the symmetrized form $d = \det(L+L')$ is relatively prime to 2.

For any dyadic (i.e. dividing 2) prime ideal $\mathfrak{p}$, let $\operatorname{Arf} L_{\mathfrak{p}}$ denote the Arf invariant of the quadratic form $x \to L(x, x) \bmod \mathfrak{p}$ over $P \otimes R/\mathfrak{p}$. For $a \in R$ and a nondyadic prime ideal $\mathfrak{p}$ we denote the quadratic residue symbol by

$$\left(\frac{a}{\mathfrak{p}}\right) = \begin{cases} 1 & \text{if} \quad a \text{ is square in } R/\mathfrak{p} \\ 0 & \text{if} \quad a \in \mathfrak{p} \\ -1 & \quad\ \ \text{otherwise}. \end{cases}$$

In the same way we denote the multiplicative extension of $\left(\frac{a}{\mathfrak{p}}\right)$ on the group of all non-dyadic ideals of R.

THEOREM. *With the above notations,*

$$\prod_{\mathfrak{p}} \operatorname{Arf} L_{\mathfrak{p}} = \left(\frac{2}{dR}\right) \tag{4}$$

where $\mathfrak{p}$ *runs through all tamely ramified dyadic prime ideals of* R *and* dR *is the principal ideal generated by* d.

In § 2 we give the necessary definitions and formulate two lemmas about Gauss sums for bilinear forms.

In § 3 we prove the theorem, using the results of § 2. The proofs of the lemmas in § 2 is given in § 4. Finally I would like to thank A. Adler, W. Pardon, and C. Weibel for useful discussions.

§ 2. Weil-Milgram quadratic reciprocity law

Let A_F denote the adele group of the field F i.e. the group of infinite vectors $(\ldots x_v \ldots)$ where v runs through all valuations of F, x_v is an element of completion F_v, and all x_v except for finite number of v's are integers. F is diagonally embedded in A_F. Let χ denote a character of A_F trivial on F. Let χ_0 be the following special character of such type. On the archimedian component v of the adele $\{x_v\} \in A_F$, χ_0 takes the value $\exp(-2\pi i \mathrm{Tr} x_v)$, and on a non-archimedian component $\exp(2\pi i \mathrm{Tr} x_v)$. Here Tr denotes the absolute trace from the v-component F_v of A_F to $Q_{\bar{v}}$ where $\bar{v}$ is a valuation of Q over which lies v. Recall that exponent of p-adic number a is defined as exponent of a component a_1 in presentation $a = a_1 + a_2$ where $a_1 \in Q$, $a_1 = p^{-n} a_1'$, $a_1' \in Z$, and $a_2 \in Z_p$. Any character χ of the type above has the form $x \to \chi_0(ax)$ for some rational a.

Let q be a quadratic form defined on the vector space V over F_v where v is one of the non-archimedian valuations of F. Suppose that L is a lattice in V such that $\chi(q(x)) = 1$ for any $x \in V$. The dual lattice L^* is defined as follows

$$L^* = \{h \in V \mid \chi(\tilde{q}(x, h)) = 1 \quad \text{for} \quad \forall x \in V\} \tag{6}$$

where

$$\tilde{q}(x, y) = q(x+y) - q(x) - q(y) \tag{7}$$

is the bilinear form associated to q. Then the correspondence $q \mapsto \gamma_v^\chi(q)$ where

$$\gamma_v^\chi(q) = \sum_{h \in L^*/L} \chi(q(h)) \Big/ \Big| \sum_{h \in L^*/L} \chi(q(h)) \Big| \tag{8}$$

defines a character of the Witt group $W(F_v)$ ([5]). (Over a field of zero characteristic we can identify the Witt group of quadratic forms with the Witt group of bilinear forms by the correspondence (7)). For an archimedian valuation v, the character γ_v^χ is defined as follows.

$$\gamma_v^\chi(q) = \exp \frac{-\pi i \sigma(q)}{4} \tag{9}$$

if F_v is R, and

$$\gamma_v^\chi(q) = 1 \tag{10}$$

if F_v is $\mathbf{C}$. ($\sigma(q)$ denotes the signature of the quadratic form q). Now suppose that q is a quadratic form over the field F. Then q defines quadratic forms q_v over all F_v and the Weil quadratic reciprocity law asserts that

$$\prod_v \gamma_v^\chi(q_v) = 1 \tag{11}$$

where v runs through all valuations of F.

If S is a symmetric bilinear form over $\mathbf{Z}$, on the lattice L such that $q(x) = S(x, x)$ is an even quadratic form, then applying (11) to $\varphi(x) = \frac{1}{2} q(x)$ and the character χ_0 of the ring $A_{\mathbf{Q}}$ defined above, one concludes that

$$e^{\frac{\pi i \sigma(q)}{4}} = \sum_{x \in L_p^{\#}/L_p} e^{2\pi i \varphi(x)} \Big/ \Big| \sum_{x \in L_p^{\#}/L_p} e^{2\pi i \varphi(x)} \Big| \tag{12}$$

where L_p is the lattice of integer vectors in the p-adic completion of L. This is the essential part of Milgram's formula ([4]).

Now let us consider properties of the character γ^χ in more detail.

Let $F_{\mathfrak{p}}$ be one of the completions of F where $\mathfrak{p}$ is a non-dyadic prime ideal.

Lemma 1. *Let q be a quadratic form over $F_{\mathfrak{p}}$. Let a be a unit in $F_{\mathfrak{p}}$. Denote by (aq) the quadratic form defined by $(aq)(x) = a \cdot q(x)$. Then*

$$\gamma_{\mathfrak{p}}^\chi(aq) = \left(\frac{a}{(\det \tilde{q}) \cdot \mathfrak{s}^{rkq}}\right) \gamma_{\mathfrak{p}}^\chi(q) \tag{13}$$

where $(-)$ is the quadratic residue symbol, $\mathfrak{s}$ is the support of the character χ, and rkq is rank of the form q.

Remark. $\det \tilde{q}$ is defined up to a square in $F_{\mathfrak{p}}$, and therefore the quadratic residue symbol in (13) is well defined.

Now we consider dyadic valuations.

Lemma 2. *Let q denote a quadratic form over a ring of integers $R_{\mathfrak{p}}$ of the dyadic field $F_{\mathfrak{p}}$ such that the determinant of the associated form $\tilde{q}$ (see (7)) is a unit in $R_{\mathfrak{p}}$. Let χ be a character of $F_{\mathfrak{p}}$ with support $R_{\mathfrak{p}}$. If $\mathfrak{p}$ is tamely ramified over $\mathbf{Q}$ then*

$$\operatorname{Arf}(q \bmod \mathfrak{p}) = \gamma_{\mathfrak{p}}^\chi(2q). \tag{14}$$

Otherwise

$$\gamma_{\mathfrak{p}}^{\chi}(2q) = 1 . \tag{15}$$

Remark. The condition on $\det \tilde{q}$ implies non-degeneracy of q at p.

§ 3. Proof of the main theorem

Note that the rank of q is even because determinant of the associated bilinear form is odd. Therefore

$$\gamma_v^{\chi}(aq) = \left(\frac{a}{(\det \tilde{q})}\right) \gamma_v^{\chi}(q) \tag{16}$$

for *any* character χ.

Now let us apply the Weil reciprocity law for the character χ with support in dyadic components equal to the integers in the corresponding ring, and to the forms q and $2q$.

We have

$$\prod_v \gamma_v^{\chi}(2q) = 1$$

$$\prod_v \gamma_v^{\chi}(q) = 1 .$$

For an archimedian components we have $\gamma_v^{\chi}(2q) = \gamma_v^{\chi}(q)$ because both depend only on the signatures. Therefore dividing those two identities, and using lemma 2 and (16) we obtain the identity (4).

Remark. Levine's lemma which in a specialization of the theorem for $R = \mathbf{Z}$ in fact follows from Milgram's formula (12). We should not worry about ramification. Therefore lemma 1 can be used for the character χ_0 and is actually a classical property of Gauss sums ([2]). Lemma 2 in this case essentially contains in [1].

§ 4. Proof of the lemmas

Proof of lemma 1. The Witt group of quadratic forms over a field of zero characteristic is generated by one-dimensional forms ([4]). Because γ^{χ} is a character of the Witt group it is enough to check the lemma for forms of one variable. Let π be a local parameter. Suppose that $q(x) = \alpha \pi^b x^2$, and $\chi(x) = \exp(2\pi i \operatorname{Tr}(\beta\pi^c x))$ where α and β are units. Suppose that the different $\mathfrak{d}$ of $F_\mathfrak{p}/\mathbf{Q}_p$ is $\mathfrak{d} = (\pi^d)$. Let n be an integer such that $2n + b + c + d > 0$. Then we can take as L the lattice (π^n). The dual lattice $L^\#$ is $(\pi^{-b-c-n+d})$. Therefore

$$\gamma^\chi(q) = \sum_{x \in (\pi^{-b-c-n-d})/(\pi^n)} \exp 2\pi i (\operatorname{Tr} \alpha\beta\pi^{c+b} x^2) \Big/ \Big| \sum_{x \in (\pi^{-b-c-n-d})/(\pi^n)} \exp 2\pi i (\operatorname{Tr} \alpha\beta\,\pi^{b+c} x^2) \Big|$$

After a change of variables $\pi^{-b-c-n-d}\, y = x$, we obtain

$$\gamma^\chi(q) = \left(\sum_{y \in R_\mathfrak{p}/(\pi^{2n+b+c+d})} \exp 2\pi i \operatorname{Tr}\left(\frac{\alpha\beta y^2}{\pi^{2n+b+c+d}\pi^d}\right)\right) \Big/ \left| \sum_{y \in R_\mathfrak{p}/(\pi^{2n+b+c+d})} \exp 2\pi i \operatorname{Tr}\left(\frac{\alpha\beta y^2}{\pi^{2n+b+c+d}\pi^d}\right)\right| \tag{17}$$

The numerator of $\gamma^\chi(q)$ is a Gauss sum of the type considered by Hecke [2]. The same arguments show that

$$\gamma^\chi(aq) = \left(\frac{a}{\pi^{2n+b+c+d}}\right)\gamma^\chi(q) \tag{18}$$

Now the support of χ is (π^{c+d}), $(\det \tilde{q})\, R_\mathfrak{p} = (\pi^b)$, and $\left(\dfrac{a}{\pi^{2n}}\right) = 1$, therefore the lemma follows.

Now let F_{2^f} denote a field of 2^f elements. Let $\tilde{\chi}$ denote a non-trivial character of the additive group of F_{2^f}. There exist a canonical choice of $\tilde{\chi}$, namely

$$\tilde{\chi}_0(x) = (-1)^{\operatorname{Tr}_{F_{2^f}|F_2}(x)} \tag{19}$$

Note that kernal of $\tilde{\chi}_0$ is an additive subgroup of elements of the form $x + x^2$.

LEMMA 3. *Let $\bar{q}$ be a non singular quadratic form defined on a vector space V over F_{2^f}. Then*

$$\gamma^{\tilde{\chi}}(\bar{q}) = \sum_{x \in V} \tilde{\chi}(\bar{q}(x)) \Big/ \Big| \sum_{x \in V} \tilde{\chi}(q(x)) \Big| \tag{20}$$

is equal to the Arf invariant of $\bar{q}$.

Proof. We follow the classical scheme in [2]. Let $x = x_1 + \pi^{n-1} x_2$, where $x_2 \in V/\pi V$ and $x_1 \in V/\pi^{n-1}$. Then

$$\sum_{x\in V/\pi^n V} \chi\left(\frac{q(x)}{\pi^n}\right) = \sum_{\substack{x_1\in V/\pi^{n-1}V\\ x_2\in V/\pi V}} \chi\left(\frac{q(x_1)+\pi^{n-1}\tilde{q}(x_1,x_2)+\pi^{2n-2}q(x_2)}{\pi^n}\right)$$

$$= N \sum_{\substack{x_1\equiv 0(\pi)\\ x_1\in V/\pi^{n-1}V}} \chi\left(\frac{q(x_1)}{\pi^n}\right) + \sum_{x_1}\chi\left(\frac{q(x_1)}{\pi}\right)\left(\sum_{\substack{x_1\not\equiv 0(\pi)\\ x_1\in V/\pi^{n-1}V\\ x_2\in V/\pi V}} \chi\left(\frac{\tilde{q}(x_1,x_2)}{\pi}\right)\right)$$

The sum in brackets is the sum of the values of the non-trivial (because det $\tilde{q}$ is unit and supp $\chi = R_p$) character, hence is equal zero. Therefore we obtain the result of the lemma because

$$\sum_{\substack{x_1\equiv 0(\pi)\\ x_1\in V/\pi^{n-1}V}} \chi\left(\frac{q(x)}{\pi^n}\right) = \sum_{x\in V/\pi^{n-1}V} \chi\left(\frac{q(x)}{\pi^{n-2}}\right)$$

Now we are ready to conclude the

Proof of lemma 2. Let e denote the ramification index of F_p over $\mathbf{Q}_2$. Thus for a character with the support R_p the dual of integer lattice V with respect to form $2q$ in the lattice $\frac{1}{\pi^e}V$. Hence

$$\gamma^\chi(2q) = \sum_{x\in\frac{1}{\pi^e}V/V} \chi(2q(x)) \Big/ \Big|\sum_{x\in\frac{1}{\pi^e}V/V} \chi(2q(x))\Big| \tag{23}$$

$$= \sum_{x\in V/\pi^e V} \chi\left(\frac{q(x)}{\pi^e}\right) \Big/ \left|\sum_{x\in V/\pi^e V} \chi\left(\frac{q(x)}{\pi^e}\right)\right|$$

If e is odd then by lemma 4, (23) is equal to

$$\sum_{x\in V/\pi V} \chi\left(\frac{q(x)}{\pi}\right) \Big/ \left|\sum_{x\in V/\pi V} \chi\left(\frac{q(x)}{\pi}\right)\right|$$

which is, by the corollary to lemma 3, the Arf invariant of q mod p. If e is even then it follows from (22) that (23) is equal 1. This concludes the proof of lemma 2.

Proof. Both Gauss sum (20) and the Arf invariant are characters of the Witt group of quadratic forms. Therefore it is enough to check the statement for binary quadratic forms. But the number of elements in $F_{2^f} \oplus F_{2^f}$ on which $\tilde{\chi}(\bar{q}(x))$ takes the value 1, is $2^{2f-1} + (\mathrm{Arf}\,\bar{q})\, 2^{f-1}$ and the number of elements where $\tilde{\chi}(\bar{q}(x))$ takes the value -1 is $2^{2f-1} - (\mathrm{Arf}\,\bar{q})\, 2^{f-1}$. Indeed it is easy to check for form of Arf invariant 1 (which is $q(\alpha, \beta) = \alpha\beta$). On the other hand form of Arf invariant -1 can be written as $q(\alpha, \beta) = \alpha^2 + \alpha\beta + s\beta^2$, where $s \neq \gamma + \gamma^2$ for any γ. ([4]). But if m is chosen in such a way that $\tilde{\chi}(x) = \tilde{\chi}_0(mx)$ ($\tilde{\chi}_0$ is defined above), we have

$$\tilde{\chi}(\alpha^2 + \alpha\beta + s\beta^2) = -\tilde{\chi}_0((s + m^2\alpha^2)(1 + m^2\beta^2))$$

i.e. number of the elements for which $\tilde{\chi}(\bar{q}(x)) = -1$ in a quadratic space with Arf invariant -1, equals the number of elements for which $\tilde{\chi}_0(q(x)) = 1$ in the space with Arf invariant 1. Therefore the lemma 3 follows.

Remark. A connection between the Arf invariant and Gauss sums was first observed in [1].

Corollary. *Let $F_\mathfrak{p}$ be a dyadic local field. Let χ denote a character of the additive group of $F_\mathfrak{p}$ with support $R_\mathfrak{p}$. Let q be an integer quadratic form on the $R_\mathfrak{p}$-module V such that the determinant of the associated bilinear form is a unit. Then*

(21)
$$\sum_{x \in V/\pi V} \chi\left(q\left(\frac{x}{\pi}\right)\right) \Big/ \left| \sum \chi\left(q\left(\frac{x}{\pi}\right)\right) \right|$$

is equal to the Arf *invariant of* q mod $\mathfrak{p}$.

Proof. The map $x \to \chi\left(\frac{x}{\pi}\right)$ defines a non-trivial character of $R_\mathfrak{p}/\pi R_\mathfrak{p}$. Therefore the expressions (20) and (21) coincide.

Lemma 4. Let χ denotes a character of the dyadic field $F_\mathfrak{p}$ with support $R_\mathfrak{p}$.

Let q denote an integer quadratic form over the $R_\mathfrak{p}$ module V such that determinant of the associated bilinear form is a unit. Then

(22)
$$\sum_{x \in V/\pi^n V} \chi\left(\frac{q(x)}{\pi^n}\right) = N^{\dim V} \sum_{x \in V/\pi^{n-2} V} \left(\frac{q(x)}{\pi^{n-2}}\right)$$

where N is the norm of the prime ideal of $R_\mathfrak{p}$.

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