ON THE TOPOLOGICAL STRUCTURE OF EVEN-DIMENSIONAL COMPLETE INTERSECTIONS

BY

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ABSTRACT. A topological connected sum decomposition into indecomposable pieces is given for complete intersections, and these pieces are described by plumbing constructions.

The principal technical results are structure theorems for the intersection form on the middle dimensional homology and the submodule of spherical classes.

1. Introduction. In this paper we describe certain algebraic manifolds up to diffeomorphism in terms of two standard constructions of differential topology, plumbing and gluing by a diffeomorphism between boundaries. As a consequence we also obtain a decomposition up to homeomorphism as a connected sum. The algebraic manifolds considered are complete intersections, that is, submanifolds of complex projective space of complex codimension r which are defined by the simultaneous vanishing of r homogeneous polynomials with linearly independent gradients. The condition on the gradients insures that each polynomial defines a nonsingular hypersurface and that their intersection is transversal.

If $X_n \subset CP_{n+r}$ is defined by polynomials of degrees $d_1, \ldots, d_r$, the multidegree is the unordered r-tuple $\mathbf{d} = (d_1, \ldots, d_r)$; the degree of X is the product $d = d_1 \cdots d_r$. The invariants n and $\mathbf{d}$ determine X up to diffeomorphism; we write $X = X_n(\mathbf{d})$. It is a consequence of the Lefschetz theorem on hyperplane sections that the homology module of X is the same as that of the complex projective space, CP_n, of the same dimension except in the middle dimension where $H_n X = Z \oplus \cdots \oplus Z$. It is natural to ask to what extent direct sum decompositions of $H = H_n X$ reflect connected sum decompositions of X.

When n is even intersection pairing equips H with a symmetric, unimodular bilinear form; we say, following [19, p. 1], that H is an inner product space over Z. Also the inclusion $i: H \to CP_{n+r}$ provides H with a distinguished element, $h = i^* x^{n/2} \cap [X]$, where $x \in H^2(CP_{n+r})$ is dual to CP_{n+r-1}, such that the orthogonal complement $h^\perp$ is equal to $\text{im}\{\pi_n X \to H_n X\}$, the set of spherical classes. Motivated by the properties of the homology of a complete intersection we define a *based inner product space* to be a pair (H, h) where H is an inner product space and h is an element of H such that

(a) h is indivisible,

(b) $h^\perp$ has even type ($u \in h^\perp$ implies $u \cdot u$ is even).

We call $d = h \cdot h$ the *degree* of (H, h). The pair (H, h) and the integer n determine completely the cohomology ring of X_n and, if $n > 2$, (H, h) is determined by the cohomology ring.

THEOREM A. *Let (H, h) be a based inner product space with $|\text{Sign } H| \leqslant \text{rank } H - 4$. Then there is a unimodular summand A of H with $h \in A$ such that $\text{rank } A \leqslant 5$.*

The condition that H be sufficiently indefinite arises because in that case the pair (H, h) is determined up to an isometry preserving the base element by the invariants rank, signature, type, and degree; see Lemma 3.7 below.

In the proof of Theorem A in §3 we will determine the minimal possible rank for A which depends on d (mainly on d mod 8) and on the type of H. In each case it is possible to choose a pair (A, h) which we can describe up to isomorphism in terms of a given basis, intersection matrix, and representation of h in terms of the basis. All this information is described by a "plumbing diagram", see §4. The pair (H, h) is determined by its rank, signature, and the pair (A, h).

For a complete intersection $X_n(\mathbf{d})$, the invariants rank and signature are computable either in terms of generating functions [8, §22.1.1] or recursion relations [8, §11.3.1]. Explicit numbers become more difficult to obtain as d and n grow. The type (even or odd) is the type of the binomial coefficient $\binom{m+s}{s}$, where s is the number of defining equations of even degree and m is half the dimension.

Cases where $h^{\perp}$ is definite so that H is definite or nearly definite (these are the cases to which Theorem A does not apply) occur only for $\mathbf{d} = (1)$, (2), or $(2,2)$ or for $n = 2$ and $\mathbf{d} = (3)$. In these cases we give explicit diagrams for H in §6. In particular for the intersection of two quadrics, $X_n(2, 2)$, we show (6.3) that $H = \Gamma_{n+4}$, the definite inner product space of rank $n + 4$ defined in [19, p. 27] or [22, V.1.4.3]. In this case there is no proper, unimodular submodule containing h.

To state our main result let V denote the $2n$-manifold with boundary obtained by plumbing eight copies of the tangent disk bundle of S^n according to the diagram E_8. We assume a general familiarity with this construction (see, for example [2, Chapter V] or [9, §8]). Let $\natural$ denote boundary connected sum.

THEOREM B. *If X is a complete intersection of complex dimension $n = 2m > 2$, then:*
$X = W_1 \cup_{\varphi} W_2$ where $\varphi: \partial W_1 \to \partial W_2$ is a diffeomorphism;
W_1 is a disk bundle over complex projective space; and
$W_2 = W \natural \alpha(S^n \times S^n - D^{2n}) \natural \beta V$ where W is obtained by plumbing a bundle over CP_m with bundles over S^n. Moreover if $X \neq X_n(2, 2)$ we may take rank $H_n W \leqslant 5$.

Explicit plumbing instructions are available. Here and below αN denotes the (boundary) connected sum of $|\alpha|$ copies of the oriented manifold (with boundary) N with opposite orientation if $\alpha < 0$. Bundles are all disk bundles associated to vector bundles with the orthogonal group as structure group. The bundles over complex projective space which occur in the construction are determined by n and $\mathbf{d}$. The bundles over S^n are stably trivial. The normal Euler classes and the mutual

intersections which determine the plumbing are given by the intersection matrix for the inner product space A of Theorem A.

Let = denote diffeomorphism and $\approx$ denote homeomorphism. Then

$$\partial(S^n \times S^n - D^{2n}) = S^{2n-1} \quad \text{and} \quad \partial V = \Sigma^{2n-1} \approx S^{2n-1}$$

where Σ^{2n-1} is a homotopy sphere. $V \cup D^{2n}$ is a closed, topological, $(n-1)$-connected $2n$-manifold with rank $H_n = 8$ and signature 8. From Theorem B we have

$$\partial W_1 = \partial W_2 = \partial W \,\#\, \alpha S^{2n-1} \,\#\, \beta \Sigma^{2n-1} = \partial W \,\#\, \Sigma' \approx \partial W.$$

Denote the composite homeomorphism by ψ.

COROLLARY C. *If X is a complete intersection of even complex dimension $n > 2$ and $X \neq X_n(2, 2)$, then $X \approx M \,\#\, \alpha(S^n \times S^n) \,\#\, \beta(V \cup D^{2n})$ where $M \approx W_1 \cup_\psi W$ and* rank $H_n M \leqslant 5$.

The proof is evident, as indicated by the picture
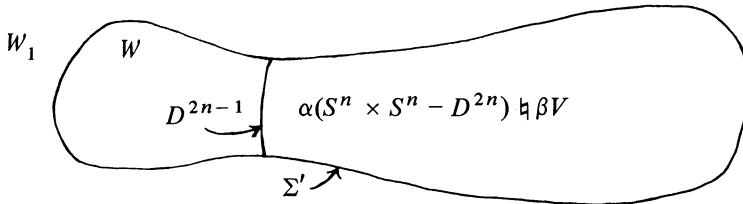
In general complete intersections with different multidegrees can nevertheless be diffeomorphic. This is classical for curves, for example, $X_1(3) = X_1(2, 2)$. Using Wall's classification of simply connected 6-manifolds and a counting argument we can show the existence of many such examples in case $n = 3$. More recently [28], [29] we have shown that there are such examples in all odd complex dimensions. For two complete intersections to be diffeomorphic they must have the same (H, h) and the same Pontryagin classes (given by certain symmetric functions in **d**). The open problem is to find additional invariants (if needed) to obtain a complete set of invariants of the diffeomorphism type or the topological type. One approach to this classification problem is through further study of the diffeomorphism φ. The one known thing along this line is that a complete intersection of low codimension, that is, satisfying $2r \leqslant n$ and $n > 2$, cannot be diffeomorphic to any other complete intersection; in fact in this case the degree and Pontryagin classes are a complete set of invariants [29, §7].

In §§2 and 5 we obtain some facts about the homology of X and in §§3, 4 and 6 give some structure theorems for based inner product spaces proving Theorem A. In §7 we reduce the proof of Theorem B to the construction in X of two subcomplexes with certain homological properties. In §8 we complete the proof in the case where H has even type. In that case rank $A = 2$ and W_1 and W are both explicitly determined bundles over CP_m. The case of odd type is completed in §9, a proof that W depends only on the pair (A, h) and an alternate description of W is given. In §10 a number of explicit examples of plumbing diagrams are given.

Throughout we must assume the complex dimension n of X is > 2 because of the failure of the Whitney embedding trick in real dimension 4. For $n = 1$ the result $X = \alpha(S^1 \times S^1)$ is very familiar. For $n = 2$ decomposition results up to homotopy are given by Milnor [16]. Recently Mandelbaum and Moishezon [14], [21] have obtained a diffeomorphism result, $X \# \overline{CP_2} = \alpha CP_2 \# \beta \overline{CP_2}$, using techniques of algebraic geometry. For $n > 2$ the decomposition $X = N \# \alpha(S^n \times S^n)$ is given in [10] for hypersurfaces and in [26] for complete intersections. Some cases of our results for hypersurfaces appeared in [12].

In the case of odd n the result is $X = M \# \alpha(S^n \times S^n)$ where rank $H_n M = 0$ or 2 depending on the Kervaire invariant; for hypersurfaces see [20], [25], or [11] and for complete intersections [27] or [3].

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2. Homology of X. In this section we collect some facts about the intersection pairing on the homology of a complete intersection X. It follows from the Lefschetz theorem on hyperplane sections [17] that the inclusion $i: X_n \to CP_{n+r}$ is an n-equivalence. In fact, there is a sequence $X_n \subset X_{n+1} \subset \cdots \subset CP_{n+r}$ where X_j is the transversal intersection of X_{j+1} with some hypersurface Y in CP_{n+r}. Let $\psi: CP_{n+r} \to CP_N$ be the Veronese embedding [23, p. 40] so that $Y = \psi^{-1}(CP_{N-1})$ for some hyperplane in CP_N. Then X_j is a hyperplane section of X_{j+1} and the Lefschetz theorem implies the inclusion $X_j \to X_{j+1}$ is a j-equivalence. The result for $X_n \to CP_{n+r}$ follows. Moreover it follows from Poincaré duality and the universal coefficient theorem that the homology of X is torsion free. So $H = H_n X$ is a free abelain group. Its rank can be computed from the Euler characteristic of X_n which is given in terms of n and $\mathbf{d}$ by a generating function due to Hirzebruch [8, p. 160]. We shall return to this question in §5.

THEOREM 2.1. *Let H be the n-dimensional homology of the complete intersection $X_n(\mathbf{d})$ of even complex dimension $n = 2m$. Then there are elements $h, y \in H$ such that*:

(1) $h \cdot h = d$, $h \cdot y = 1$, *and* $y \cdot y \equiv \binom{m+s}{m}$ mod 2 *where s is the number of even entries in $\mathbf{d}$.*

(2) $h^\perp$ *has even type ($u \cdot h = 0$ implies $u \cdot u$ is even).*

(3) *For $u \in H$, $u \cdot h = 0$ implies u is represented by an embedded $S^n \subset X_n$.*

(4) *For $u \in H$, $u \cdot h = 1$ implies u is represented by an embedded $CP_n \subset X_n$.*

COROLLARY 2.2. *H has even type if and only if $\binom{m+s}{m}$ is even.*

PROOF OF 2.1. Let $x \in H^2(CP_{n+r})$ be dual to CP_{n+r-1}, so $x \cap [CP_{n+r}] = [CP_{n+r-1}]$. Let $h = (i^* x^m) \cap [X]$ where $i: X \hookrightarrow CP_{n+r}$. If d is the degree of X, then $i_*[X] = dx^r \cap [CP_{n+r}]$, so $h \cdot h = (i^* x^n) \cap [X] = x^n \cap i_*[X] = d$.

Since i is an n-equivalence by Lefschetz, the map $i_\#: [CP_m, X] \to [CP_m, CP_{n+r}]$ is onto. Therefore there is a map $j: CP_m \to X$ so that $i \circ j$ is homotopic to a linear embedding. Further by replacing j by a homotopic map we may assume j is an embedding by a result of Haefliger [6]. Let $y = j_*[CP_m]$.

Let $D: H_n X \to H^n X$ denote Poincaré duality, so $(Dv) \cap [X] = v$. Then

$$h \cdot y = \left(i^* x^m \cup Dj_*[CP_m] \right) \cap [X] = i^* x^m \cap j_*[CP_m]$$
$$= j_*\left(j^* i^* x^m \cap [CP_m] \right) = 1.$$

A short argument [26, Lemma 2] using the Hurewicz ladder shows

$$\text{im}\{\pi_n X \to H_n X\} = h^\perp = \{u \in H: u \cdot h = 0\}.$$

By [6] it follows that any $u \in h^\perp$ is represented by an embedded sphere $S^n \subset X_n$.

Let $v = y + u \in y + h^\perp$. Since y is represented by $j: CP_m \hookrightarrow X_n$ and u is represented by some map of S^n to X, v is represented by a map of $CP_m = CP_m \# S^n \to X$, which, by [6], we may take to be an embedding.

Given $S^n \overset{e}{\hookrightarrow} X \overset{i}{\hookrightarrow} CP_{n+r}$, since the normal bundle $\nu(i)$ is the restriction of a bundle over CP_{n+r} and $i \circ e$ is null-homotopic, $\nu(e)$ is stably trivial. It follows that if $u = e_*[S^n]$, $u \cdot u = \chi\nu(e) \equiv w_n \nu(e) = 0 \bmod 2$, so $h^\perp$ has even type.

Finally the embeddings $CP_m \overset{j}{\hookrightarrow} X \overset{i}{\hookrightarrow} CP_{n+r}$ give the bundle equation:

LEMMA 2.3. $\tau CP_m \oplus \nu(j) \oplus j^* \nu(i) = j^* i^* \tau CP_{n+r}$, and $\nu(i) = i^*(\gamma^{d_1} \oplus \cdots \oplus \gamma^{d_r})$, where γ^k is the k-fold tensor power of the Hopf bundle.

If $d_1, \ldots, d_s$ are even and $d_{s+1}, \ldots, d_r$ are odd, working in $H^*(CP_m; Z/2)$ we have

$$(1 + x)^{m+1} W(\nu(j))(1 + x)^{r-s} = (1 + x)^{n+r-1},$$

hence $W(\nu(j)) = (1 + x)^{m+s}$, so

$$w_{2m}(\nu(j)) \cap [CP_m] = \binom{m + s}{m} \bmod 2.$$

It follows that

$$y \cdot y = \chi\nu(j) \equiv \binom{m + s}{m} \bmod 2.$$

This completes the proof of 2.1.

The normal bundles of these embedded spheres and complex projective spaces will be the bundles specified in our plumbing constructions in proving Theorem B. We conclude this section by showing they are determined by m, $\mathbf{d}$, and by the self-intersection of the homology class represented.

In the case $S^n \hookrightarrow X_n$ representing a class u, the normal bundle is stably trivial and so is determined by its Euler class which is dual to $u \cdot u$.

If $j: CP_m \hookrightarrow X_n$ with $i \circ j: CP_m \hookrightarrow CP_{n+r}$ homotopic to the linear embedding, then $\nu(j)$ is determined by m and $\mathbf{d}$ as a stable bundle according to the bundle equation (Lemma 2.3). Its normal Euler class is given by $y \cdot y$ where $y = j_*[CP_m]$. In general there is the following.

LEMMA 2.4. *Two oriented, 2m-dimensional vector bundles over a 2m-dimensional complex are equivalent if and only if they are stably equivalent and have the same Euler class.*

PROOF. Let E be the fibered product of the maps ρ_2 and w_n in the diagram ($n = 2m$):

$$
\begin{array}{ccccc}
BSO(n) & \overset{\chi}{\nrightarrow} & E & \to & K(Z, n) \\
 & & \downarrow & & \downarrow{\scriptstyle\rho_2} \\
 & & BSO(n+1) & \overset{w_n}{\to} & K(Z/2, n)
\end{array}
$$

We claim $BSO(n) \to E$ is an $n + 1$ equivalence. Then $[X, BSO(x)] \overset{=}{\to} [X, E]$ for $\dim X \le n$; hence an oriented n-dimensional bundle over X is equivalent to a stable bundle ξ and a choice of χ compatible with $w_n(\xi)$.

Now E is the total space of the fibration induced by w_n from $K(Z, n) \overset{2}{\to} K(Z, n) \overset{\rho_2}{\to} K(Z/2, n)$ so we have

$$
\begin{array}{ccccc}
S^n & \to & K(Z, n) & \overset{=}{\to} & K(Z, n) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle 2} \\
BSO(n) & \to & E & \to & K(Z, n) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle\rho_2} \\
BSO(n+1) & \overset{=}{\to} & BSO(n+1) & \overset{w_n}{\to} & K(Z/2, n)
\end{array}
$$

The exact homotopy ladder shows $BSO(n) \to E$ is an $n + 1$ equivalence.

3. Based inner product spaces. In this section we give a direct sum decomposition of the middle dimensional homology H of $X_{2m}(\mathbf{d})$. We will find a summand A of H which is unimodular and contains h and which has minimal rank subject to these two conditions. It follows that there is an orthogonal splitting, $H = A \oplus B$, which will give rise to our geometric splitting of X.

By (2.1) the homology of a complete intersection satisfies the following.

DEFINITION 3.1. A *based inner product space* is a pair (H, h) where H is an inner product space and h is an element of H such that

(a) h is indivisible,

(b) the inner product restricted to $h^\perp$ has even type.

The norm of h, $h \cdot h$, is called the *degree* of (H, h).

We fix a notation for two particular inner product spaces. Let U be the hyperbolic plane which is free of rank 2 with intersection matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$; in the notation of [19, p. 3], $U = \langle\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\rangle$. Let Γ_8 be the inner product space defined as a subspace of Q^8 in [19, p. 27] or [22, V.1.4.3] or as $\langle M_0 \rangle$ where M_0 is the 8×8 matrix defined in [2, p. 120]. Thus Γ_8 has rank 8, signature 8, and even type. (These properties characterize Γ_8.)

A more detailed statement of Theorem A in case $d \ge 0$ is given by

THEOREM 3.2. *Let (H, h) be a based inner product space with $d = h \cdot h > 0$ and $|\mathrm{Sign}\, H| \le \mathrm{rank}\, H - 4$. Then there is a unimodular summand $A \subset H$ with $h \in A$ and integers α and β such that $H = A \oplus \alpha U \oplus \beta\Gamma_8$ is an orthogonal direct sum and such that:*

(a)   *if H has even type,* $\mathrm{rank}\, A = 2$;

(b)   *if H has odd type and*

$$d = 1, \qquad \text{rank } A = 1;$$

$$d \equiv 0 \bmod 8, \qquad \text{rank } A = 2;$$

$d \equiv 2 \bmod 8$ *and all odd prime*
divisors of d are $\equiv 1 \bmod 4$, $\qquad \text{rank } A = 2;$

$$d \equiv 1, 3, \text{ or } 7 \bmod 8, \qquad \text{rank } A = 3;$$

$$d \equiv 2, 4, \text{ or } 6 \bmod 8, \qquad \text{rank } A = 4;$$

$$d \equiv 5 \bmod 8, \qquad \text{rank } A = 5.$$

Conversely these ranks are minimal.

For example if rank $A = 2$, then either H has even type, $d \equiv 0 \bmod 8$, or $d \equiv 2 \bmod 8$ and d is a sum of two squares.

To prove 3.2 we will construct particular minimal models for the space A and then use a result of C. T. C. Wall on the group of isometries of an inner product space to embed A in a given H (see 3.7).

In each case except $d \equiv 5$ the inner product space A is determined up to isomorphism by its rank, type, and by $d \bmod 8$. In case $d \equiv 5$ there are two possibilities of rank 5, either signature 5 or signature -3.

The pair (A, h) however is not unique up to isomorphism. For an easy example in $A = \langle 1 \rangle \oplus \langle 1 \rangle$ both $11e_1 + 3e_2$ and $9e_1 + 7e_2$ are possible base elements of the same norm but no automorphism of A takes one to the other. For an indefinite example in $\langle 1 \rangle \oplus \langle -1 \rangle$ take $5e_1 - e_2$ and $7e_1 - 5e_2$. The homology of $X_n(2, 2)$ provides an example of large rank for $n \equiv 2 \bmod 4$ (see §6). The fact that $|\text{Sign}| > \text{rank} - 4$ is an essential feature of these examples (see 3.7).

LEMMA 3.3. *Let H be an inner product space of odd type and let $h \in H$ be indivisible. Then $h^\perp$ has even type if and only if h is characteristic.*

PROOF. That h is characteristic means for any $x \in H$, $x \cdot x \equiv x \cdot h \bmod 2$, so $x \in h^\perp$ implies $x \cdot x$ is even. Conversely, since h is indivisible and H is unimodular there is a $y \in H$ with $y \cdot h = 1$. Then for any $x \in H$, $x - (x \cdot h)y \in h^\perp$. If H has odd type and $h^\perp$ has even type, $y \cdot y$ must be odd. Therefore $x \cdot x \equiv (x \cdot h)^2 y \cdot y \equiv x \cdot h \bmod 2$, so h is characteristic.

If H has even type, 0 is characteristic, but an indivisible class h cannot be characteristic.

COROLLARY 3.4. *If H has odd type, $\text{Sign } H \equiv h \cdot h \bmod 8$. If H has even type, $\text{Sign } H \equiv 0 \bmod 8$ and $h \cdot h$ is even.*

This follows from the lemma of van der Blij [**19**, p. 24]. Note this corollary relates the degree and signature of a complete intersection.

LEMMA 3.5. *If (H_1, h_1) and (H_2, h_2) are pairs with h_1 and h_2 characteristic, then so is $(H_1 \oplus H_2, h_1 + h_2)$.*

PROOF. Let $x = u + v \in H_1 \oplus H_2$; then

$$x \cdot x = u \cdot u + v \cdot v \equiv u \cdot h_1 + v \cdot h_2 (\bmod 2) = (u + v) \cdot (h_1 + h_2).$$

The lemmas permit us to assert that the direct sum $(H_1, h_1) \oplus (H_2, h_2) = (H_1 \oplus H_2, h_1 + h_2)$ is a based inner product space in many cases. This is true if (i) both pairs are based inner product spaces of the same type or (ii) if (H_1, h_1) is a based inner product space, H_2 has even type, and $h_2 = 0$. But note that if (H_1, h_1) is a based inner product space of odd type and (H_2, h_2) is one of even type with $h_2 \neq 0$, then $(h_1 + h_2)^\perp$ will not have even type.

LEMMA 3.6. *If (H, h) is a based inner product space, $(h^\perp)^\# / h^\perp = Z/d$ where $h \cdot h = d$ and $\#$ denotes $\mathrm{Hom}(\ , Z)$.*

We shall need this only incidentally; a proof is given in [**10**, Proposition 9.5].

The following classification of sufficiently indefinite based inner product spaces is our main tool.

LEMMA 3.7. *Two based inner product spaces with the same rank, signature, type, and degree and which satisfy $|\mathrm{Sign}| \leqslant \mathrm{rank} - 4$ are isometric by an isometry which preserves the base element.*

PROOF. Indefinite inner product spaces of the same rank, signature, and type are isometric [**19**, p. 25]. Under the hypothesis that $|\mathrm{Sign}| \leqslant \mathrm{rank} - 4$, a result of Wall [**24**] shows the group of isometries of a space of even type acts transitively on indivisible vectors of a given norm and the group of isometries of a space of odd type acts transitively on indivisible, characteristic vectors of a given norm. The lemma follows.

We are now ready for the proof of Theorem 3.2.

PROOF OF 3.2(a). Since H has even type, $h \cdot h$ is even, say $h \cdot h = 2a$. Let U be the hyperbolic plane with basis e, f and intersection matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Let $h_1 = ae + f$. Then h_1 is indivisible with norm $2a$. Then there are integers α and β such that the based inner product space $(U, h_1) \oplus \alpha(U, 0) \oplus \beta(\Gamma_8, 0)$ has the same rank, signature, type, and degree as (H, h). By 3.7 they are isometric and hence $h \in H$ is contained in a unimodular summand of rank 2. This of course is minimal.

Next we consider based inner product spaces of odd type and give examples of minimal rank for each degree mod 8. In each case the proof of 3.2(b) will be completed by a similar application of 3.7.

Case $d = 8a$. Let $A_0 = \langle 1 \rangle \oplus \langle -1 \rangle$ with basis u, v. Let $h_0 = (2a + 1)u + (2a - 1)v$. Then h_0 is indivisible of norm $8a$. If $x = ru + sv$, then $x \cdot x = r^2 - s^2 \equiv r + s \bmod 2$ and $x \cdot h_0 \equiv r + s \bmod 2$ so h_0 is characteristic. Rank 2 is minimal since $\mathrm{rank}\, H \equiv \mathrm{sign}\, H \bmod 2$. It follows there is an isometry of (A_0, h_0) into (H, h).

Case $d = 8a + 1$. For $d = 1$, take $\langle 1 \rangle$ with h as basis. In general take $(A_0, h_0) \oplus (\langle 1 \rangle, x_1)$; here $x_1 \cdot x_1 = 1$. By 3.5 this is a based inner product space of odd type and degree $8a + 1$. Since $\mathrm{Sign}\, H \equiv 1 \bmod 8$ and H is unimodular, rank 3 is minimal for $a \neq 0$.

Case $d = 8a + 2$. For $d = 2$ take $(\langle 1 \rangle, x_1) \oplus (\langle 1 \rangle, x_2)$. In general we may take $(A_0, h_0) \oplus (\langle 1 \rangle, x_1) \oplus (\langle 1 \rangle, x_2)$. A rank 2 example must be positive definite (since of signature 2), hence of the form $\langle 1 \rangle \oplus \langle 1 \rangle$ with basis u, v. Then if $h = ru + su$, $h \cdot h = d = r^2 + s^2$. Since h is indivisible, r and s must relatively prime. This is

possible if and only if $4 \nmid d$ and all odd prime divisors of d are $\equiv 1 \bmod 4$ (see remark below). Since $d \equiv 2 \bmod 8$, both r and s are odd which implies $h = ru + sv$ is characteristic.

Case $d = 8a + 3$. The rank of any example must be at least 3 and a rank 3 example must be positive definite and hence of the form $A_3 = \langle 1 \rangle \oplus \langle 1 \rangle \oplus \langle 1 \rangle$ [19, p. 19]. By a result of Lagrange [5, p. 261], and see remark below, any number d neither $\equiv 7 \bmod 8$ nor divisible by 4 can be written as the sum of three squares with no common factor. Then $h_3 = a_1 e_1 + a_2 e_2 + a_3 e_3$ is indivisible and has the correct norm $h_3 \cdot h_3 = \Sigma a_i^2 = d$. The possible squares mod 8 are 0, 1, and 4, hence each a_i must be odd so h_3 is characteristic.

REMARK 3.8. The characterization of those integers expressible as the sum of two relatively prime squares is fairly well known and easy to explain, for example using prime factorization in the Gaussian integers $Z[i]$. Alternatively, the equation $x^2 \equiv -1 \bmod d$ is solvable if and only if $4 \nmid d$ and all odd prime divisors of d are $\equiv 1 \bmod 4$. Say $cd = x^2 + 1$. In that case the matrix $\left(\begin{smallmatrix} d & x \\ x & c \end{smallmatrix}\right)$ defines a unimodular, positive definite bilinear form space and d is represented by an indivisible element. But this form is equivalent to $\langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \rangle$ so $d = r^2 + s^2$ with $(r, s) = 1$. Conversely an indivisible element $re_1 + se_2$ can be completed to a basis whose intersection matrix $\left(\begin{smallmatrix} d & x \\ x & c \end{smallmatrix}\right)$ has determinant $+1$.

Dirichlet's proof [5, pp. 263, 264] of Lagrange's result on three squares uses the same approach. We present the case for $d = 8a + 3$. The matrix

$$\begin{bmatrix} d & 0 & 1 \\ 0 & b & x \\ 1 & x & c \end{bmatrix}$$

will define a unimodular, positive definite form (and hence d is equal to a sum of three squares with no common factor) if $d > 0$, $b > 0$, and the determinant $= +1$. Setting $\Delta = bc - x^2$, the condition on the determinant is $d\Delta - b = 1$. We wish to find Δ and b so that $x^2 \equiv -\Delta \bmod b$ has a solution. Let $\Delta = 8t + 1$ and choose t so that $\frac{1}{2}b = \frac{1}{2}(d\Delta - 1) = 4dt + \frac{1}{2}(d - 1)$ is a prime $p > 0$ by Dirichlet's theorem on primes in arithmetic progressions [22, pp. 103ff.]. Then $b = 2p$ and since $-2 \equiv 4p \bmod \Delta$, $1 = (-2/\Delta) = (p/\Delta) = (\Delta/p) = (-\Delta/p)$ so $-\Delta$ is a quadratic residue mod b. Then $bc = x^2 + \Delta$ and $d\Delta - b = 1$ by definition of p. This completes the proof.

Case $d = 8a + 4$. Sign $H \equiv 4 \bmod 8$, so the rank must be at least 4, therefore $(A_3, h_3) \oplus (\langle 1 \rangle, x_1)$ is an example of minimal rank.

Case $d = 8a + 5$. A rank 3 example would need to be negative definite but that is impossible with $h \cdot h \geqslant 0$; hence H must have rank at least 5. We may take $(A, h) = (A_3, h_3) \oplus (\langle 1 \rangle, x_1) \oplus (\langle 1 \rangle, x_2)$. In this case however we may also take $A = \langle 1 \rangle \oplus 4\langle -1 \rangle$. To prove this we must produce an indivisible, characteristic $h \in A$ with $h \cdot h = d$. Let $a_1 = \frac{1}{2}(d + 1)$ and $a_2 = \frac{1}{2}(d - 3)$. Then $a_1^2 - a_2^2 = 4d - 4$ so we must find a_3, a_4, a_5 such that $a_3^2 + a_4^2 + a_5^2 = 3d - 4$. Since this number is $\equiv 3 \bmod 8$ we may choose relatively prime values for a_3, a_4, a_5 satisfying the condition. As in the case $d \equiv 3$ these a's must be odd as are a_1 and a_2 (since $d \equiv 5 \bmod 8$). Hence $h = a_1 e_1 + \cdots + a_5 e_5$ is characteristic and indivisible as required.

Case $d = 8a - 2$. Take $(A_0, h_0) \oplus (\langle -1 \rangle, x_1) \oplus (\langle -1 \rangle, x_2)$. This gives an example of rank 4. Rank 2 is impossible since the space may not be negative definite.

Case $d = 8a - 1$. Take $(A_0, h_0) \oplus (\langle -1 \rangle, x_1)$. This has rank 3 which is minimal. This completes the proof of (3.2).

To complete the proof of Theorem A it is easy to check that if H has odd type and $d < 0$ the corresponding table is

$d = -1$,	rank $A = 1$,
$d \equiv 0 \bmod 8$,	rank $A = 2$,
$d \equiv 6 \bmod 8$ and all odd prime	
divisors of $-d$ are $\equiv 1 \bmod 4$,	rank $A = 2$,
$d \equiv 1, 5, \text{ or } 7 \bmod 8$,	rank $A = 3$,
$d \equiv 2, 4, \text{ or } 6 \bmod 8$,	rank $A = 4$,
$d \equiv 3 \bmod 8$,	rank $A = 5$.

We conclude this section by describing the results in the context of Witt groups (see [19, pp. 12ff.]). We define a split based inner product space (S, s) to be an inner product space S with a direct summand N such that $N = N^{\perp}$ together with an element $s \in S$ of norm 0. Two based inner product spaces of the same type (H_1, h_1) and (H_2, h_2) are Witt equivalent if there are split spaces of that type such that $(H_1, h_1) \oplus (S_1, s_1) = (H_2, h_2) \oplus (S_2, s_2)$. The equivalence classes then form the two Witt groups WH^{even} and WH^{odd}. Each of these groups is isomorphic to $Z \oplus Z$. The isomorphism is given by $(H, h) \mapsto (d/2, \sigma/8)$ in the even case and by $(H, h) \mapsto (d, (\sigma - d)/8)$ in the odd case where σ denotes the signature of H. According to 3.7, two based inner product spaces with $r - |\sigma| \geqslant 4$ are isomorphic if and only if they have the same rank, type, and Witt class. Theorem 3.2 gives elements of minimal rank in each Witt class (after some obvious remarks are added to cover negative degrees).

4. Simply connected intersection diagrams. In this section we give another direct sum decomposition $H = A \oplus B$ in the case where H has odd type. This will lead to a geometric decomposition of X as in Theorem B in which the rank of A will be somewhat larger than the minimal examples of §3 but for which we shall be able to describe the plumbing construction of the manifold W completely.

To a basis $e_1, \ldots, e_n$ of a bilinear form space A we associate a diagram D (cf. [9, p. 58]). D will be a graph consisting of vertices and edges each with integer labels. The vertices correspond to the basis elements $e_1, \ldots, e_n$. The label for the vertex j is the integer $e_j \cdot e_j$. To each pair, (i, j) with $e_i \cdot e_j \neq 0$ there corresponds an edge connecting i and j with label $e_i \cdot e_j$. Also we adopt the convention that an edge labelled $+1$ can be written with no label and an edge labelled -1 can be written $\cdot \overset{-}{\text{——}} \cdot$. For example, $\overset{0}{\cdot} \text{——} \overset{0}{\cdot}$ is the diagram of the hyperbolic plane $\langle \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right) \rangle$. The intersection matrix is completely determined by the diagram.

For a based inner product space (H, h) we will take basis elements only from the subspace $h^{\perp}$ or from the coset $\{ y \in H : y \cdot h = 1 \}$. We use a dot $\cdot$ for a vertex corresponding to an element in $h^{\perp}$ and a cross $\times$ for a vertex corresponding to an element y with $y \cdot h = 1$. The pair (H, h) is completely determined by its diagram.

Thus for a complete intersection the $\cdot$'s can be represented by embedded spheres and the $\times$'s by embedded CP_m's.

With this notation the rank 2 space of 3.2(a) is given by

$$\underset{\times}{0} \quad\rule{2cm}{0.4pt}\quad \underset{\bullet}{-2a}$$
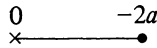
We will construct based inner product spaces denoted (A_b, h_b) for $b \geqslant 0$ and (A_{-b}, h_{-b}) for $b > 0$ of rank $6 + b$ and with a basis corresponding to the diagrams
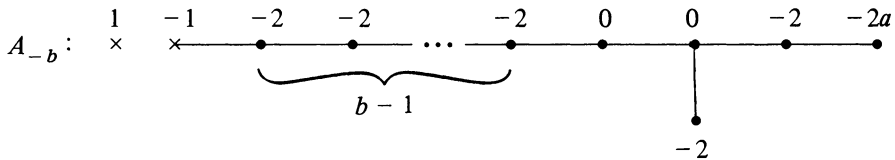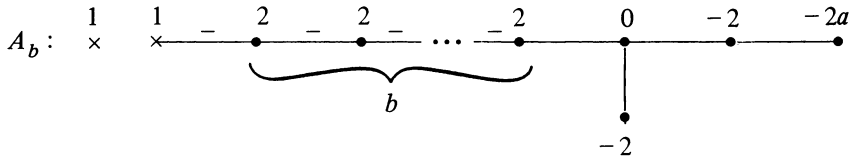
Let σ^+ and σ^- be the number of positive and negative squares in a diagonalization of H over the rational numbers. Thus rank $H = \sigma^+ + \sigma^-$ and the signature $\sigma = \sigma^+ - \sigma^-$.

THEOREM 4.1. *Let (H, h) be a based inner product space of odd type and of degree d. If $d = 8a + b$, $b \geqslant 0$, and $\sigma^+ \geqslant 3 + b$, $\sigma^- \geqslant 3$, then $A_b \subseteq H$ with $h_b = h$. If $d = 8a - b$, $b > 0$, and $\sigma^+ \geqslant 3$, $\sigma^- \geqslant 3 + b$, then $A_{-b} \subseteq H$ with $h_{-b} = h$.*

PROOF. The proof follows the pattern of §3. We construct a model pair (A_0, h_0) and then embed it in (H, h) using 3.7. Let A_0 be free of rank 6 on the generators $y_1, y_2, e_1, e_2, e_3, e_4$ with unimodular intersection matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2a & 0 \\ 0 & 0 & 1 & 0 & 0 & -2 \end{bmatrix}.$$

Thus denoting the y's by $\times$ and the e's by $\cdot$, A_0 has the prescribed diagram. Let

$$h_0 = y_1 + (8a - 1)y_2 - 2(4a - 1)e_1 - 4ae - 2e_3 - (4a - 1)e_4.$$
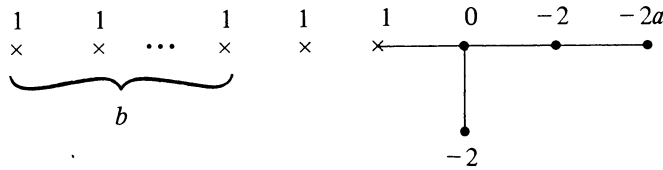
Then h_0 is indivisible and it is easy to compute $h_0 \cdot y_i = 1$ for $i = 1, 2$ and $h_0 \cdot e_i = 0$ for $1 \leqslant i \leqslant 4$. Also $h_0 \cdot h_0 = 8a$.

Finally a straightforward computation shows that h_0 is characteristic. Indeed if $x = b_1 y_1 + b_2 y_2 + c_1 e_1 + \cdots + c_4 e_4$, then $x \cdot x \equiv b_1 + b_2 \bmod 2$, and $h_0 \cdot x \equiv b_1 + b_2 \bmod 2$.

It follows from 3.3 that (A_0, h_0) is a based inner product space of odd type and degree $8a$. Diagonalizing we compute Sign $A_0 = 0$, so $\sigma^+(A_0) = 3$ and $\sigma^-(A_0) = 3$.

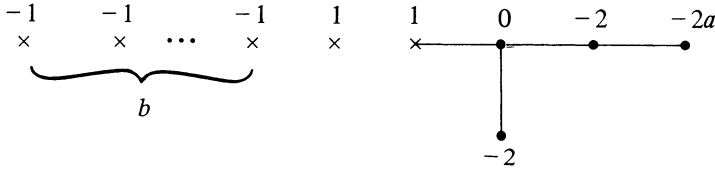Given (H, h) as in 4.1, there are integers α and β such that the based inner product space $(A_0, h_0) \oplus \alpha(U, 0) \oplus \beta(\Gamma_8, 0)$ has the same rank, signature, type, and degree as (H, h). By 3.7 it is isometric with (H, h) and hence h is contained in a unimodular summand isometric to A_0.

If, for $b > 0$, we set $(A_b, h_b) = (\langle 1 \rangle, x_1) \oplus \cdots \oplus (\langle 1 \rangle, x_b) \oplus (A_0, h_0)$, then (A_b, h_b) has odd type, degree $8a + b$, rank $6 + b$, and signature b. Recall $h_b = x_1 + \cdots + x_b + h_0$, so $h_b \cdot x_i = 1$. The corresponding diagram is
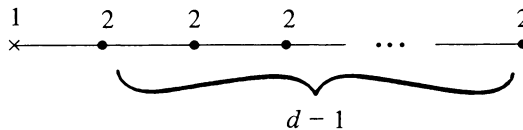
$$
\begin{array}{ccccccccc}
1 & 1 & & 1 & 1 & 1 & 0 & -2 & -2a \\
\times & \times & \cdots & \times & \times & \times\!\!-\!\!-\!\!-\!\!-\bullet\!\!-\!\!-\!\!-\bullet\!\!-\!\!-\!\!-\bullet \\
\underbrace{\qquad\qquad\qquad\qquad}_{b} & & & & & | & & \\
& & & & & \bullet & & \\
& & & & & -2 & &
\end{array}
$$

With respect to the new basis $y_1, x_1, x_2 - x_1, x_3 - x_2, \ldots, x_b - x_{b-1}, y_2 - x_b, e_1, \ldots, e_4$, the diagram is as promised above.

Finally we set $(A_{-b}, h_b) = (\langle -1 \rangle, x_1) \oplus \cdots \oplus (\langle -1 \rangle, x_b) \oplus (A_0, h_0)$ to obtain a space of odd type, degree $8a - b$, rank $b + 6$, and signature $-b$. Here $h_{-b} = x_1 + \cdots + x_b + h_0$ and $h_{-b} \cdot (-x_i) = +1$. Taking the basis $-x_1, \ldots, -x_b$, $y_1, y_2, e_1, \ldots, e_4$ the diagram is

$$
\begin{array}{ccccccccc}
-1 & -1 & & -1 & 1 & 1 & 0 & -2 & -2a \\
\times & \times & \cdots & \times & \times & \times\!\!-\!\!-\!\!-\!\!-\bullet\!\!-\!\!-\!\!-\bullet\!\!-\!\!-\!\!-\bullet \\
\underbrace{\qquad\qquad\qquad\qquad}_{b} & & & & & | & & \\
& & & & & \bullet & & \\
& & & & & -2 & &
\end{array}
$$

With respect to the new basis $y_1, -x_1, -x_2 + x_1, \ldots, -x_b + x_{b-1}, y_2 + x_b$, $e_1, \ldots, e_4$ the diagram is as promised above.

EXAMPLE 4.2. Another quite simple plumbing diagram

$$
\begin{array}{ccccccc}
1 & 2 & 2 & 2 & & & 2 \\
\times\!\!-\!\!-\!\!-\bullet\!\!-\!\!-\!\!-\bullet\!\!-\!\!-\!\!-\bullet\!\!-\!\!-\!\!-\cdots\!\!-\!\!-\!\!-\bullet \\
& \underbrace{\qquad\qquad\qquad\qquad}_{d-1} & & & & &
\end{array}
$$

corresponds to a based inner product space (A, h) which has odd type, rank d, and degree d, and is positive definite, in fact equivalent to $d\langle 1 \rangle$. Given (H, h) of odd type and degree d, if $\sigma^+(H) \geqslant d$ and rank $H - |\text{Sign } H| \geqslant 4$, then (A, h) embeds in (H, h).

5. The signature and rank of H.

In this section we show that the decomposition results of §3 can be applied to the homology of any complete intersection of even complex dimension $n \geqslant 4$ except projective space, the quadric, and the complete intersection of two quadrics. In fact we will show in §6 that these exceptional varieties admit no connected sum decomposition. We also show that the results of 4.1 apply with the additional exception of $X_4(3)$.

Let $\sigma_n(\mathbf{d})$ and $b_n(\mathbf{d})$ denote the signature and nth Betti number of $X_n(\mathbf{d})$.

THEOREM 5.1. *For* $n \geqslant 2$, $|\sigma_n(\mathbf{d})| \leqslant b_n(\mathbf{d}) - 4$ *unless* $\mathbf{d} = (1)$, (2), *or* $(2, 2)$ *or unless* $n = 2$ *and* $\mathbf{d} = (3)$.

COROLLARY 5.2. *If* $2m \geqslant 4$ *and* $\mathbf{d} \neq (2, 2)$, *then there is a unimodular submodule* A *of* $H_{2m}(X)$ *with* $h \in A$ *and* rank $A \leqslant 5$.

This follows from §3.

Recall $\sigma_n^+(\mathbf{d})$ and $\sigma_n^-(\mathbf{d})$ are the number of positive and negative squares, respectively, in a representation of the intersection form as a sum of squares. For §4 we need

THEOREM 5.3. *For* $n \geqslant 4$ *and* $\mathbf{d} \neq (1)$, (2), *or* $(2, 2)$ *we have* $\sigma_n^{\pm}(\mathbf{d}) \geqslant 7$ *except for* $n = 4$ *and* $\mathbf{d} = (3)$ *or* $(2, 2, 2)$. *For* $\mathbf{d} = (2, 2, 2)$, $\sigma_4^+ = 38$ *and* $\sigma_4^- = 6$.

To prove these results we use the Hirzebruch recursion relation for the T_y-genus to study the behavior of $\sigma_n^{\pm}(\mathbf{d})$ (cf. [13]).

LEMMA 5.4. *The functions* $\sigma_n^+(\mathbf{d})$ *and* $\sigma_n^-(\mathbf{d})$ *are monotonically increasing functions of each variable* d_j.

PROOF. It suffices to show $\sigma_n^{\pm}(\mathbf{d}, u + 1) \geqslant \sigma_n^{\pm}(\mathbf{d}, u)$ where $(\mathbf{d}, u) = (d_1, \ldots, d_r, u)$.

Let $e_n(\mathbf{d})$ denote the Euler characteristic and $b_n(\mathbf{d})$ the middle Betti number of $X_n(\mathbf{d})$ for $n \geqslant 0$ and 0 for $n < 0$. Then

(1) $\qquad e_n = n + b_n$ and $e_{n-1} = n - b_{n-1}$ for even $n \geqslant 0$.

By [8, Theorem 11.3.1] we have

(2) $\qquad \sigma_n(\mathbf{d}, u + 1) = \sigma_n(\mathbf{d}, u) + \sigma_n(\mathbf{d}) - \sigma_{n-2}(\mathbf{d}, u, u + 1)$,

(3) $\qquad e_n(\mathbf{d}, u + 1) = e_n(\mathbf{d}, u) + e_n(\mathbf{d}) - 2e_{n-1}(\mathbf{d}, u) + e_{n-2}(\mathbf{d}, u, u + 1)$.

Combining (1) and (3) we obtain for $n \geqslant 1$,

(4) $\quad b_n(\mathbf{d}, u + 1) = b_n(\mathbf{d}, u) + b_n(\mathbf{d}) + 2b_{n-1}(\mathbf{d}, u) + b_{n-2}(\mathbf{d}, u, u + 1) - 2$.

Note that (4) holds for both odd and even values of n.

Using the relations $b_n = \sigma_n^+ + \sigma_n^-$ and $\sigma_n = \sigma_n^+ - \sigma_n^-$ we find, for $n \geqslant 2$,

(5) $\quad \sigma_n^+(\mathbf{d}, u + 1) = \sigma_n^+(\mathbf{d}, u) + \sigma_n^+(\mathbf{d}) + b_{n-1}(\mathbf{d}, u) + \sigma_{n-2}^-(\mathbf{d}, u, u + 1) - 1$,

(6) $\quad \sigma_n^-(\mathbf{d}, u + 1) = \sigma_n^-(\mathbf{d}, u) + \sigma_n^-(\mathbf{d}) + b_{n-1}(\mathbf{d}, u) + \sigma_{n-2}^+(\mathbf{d}, u, u + 1) - 1$.

For $n = 0$, $b_0(\mathbf{d}) = \sigma_0^+(\mathbf{d}) = d = \Pi d_j$ and $\sigma_0^-(\mathbf{d}) = 0$. In general $\sigma_n^+(\mathbf{d}) \geqslant 1$ since the homology class h has positive norm. Thus (5) and (6) imply Lemma 5.4.

For the quadric, $\mathbf{d} = (2)$, and the intersection of two quadrics, $\mathbf{d} = (2, 2)$, (5) and (6) allow us to compute:

	$n \equiv 0 \bmod 4$	$n \equiv 2 \bmod 4$
$\sigma_n^+(2)$	2	1
$\sigma_n^-(2)$	0	1
$\sigma_n^+(2, 2)$	$n + 4$	1
$\sigma_n^-(2, 2)$	0	$n + 3$

In these cases the intersection form is definite on the subspace $h^\perp$ of spherical classes. The values of σ_n and b_n are easily determined.

To prove 5.1 we must show both $\sigma^\pm \geqslant 2$ and for 5.3 both $\sigma^\pm \geqslant 7$. We check some initial cases and then apply Lemma 5.4.

Using $b_{n-1}(2, 2) = n$ for n even (and $\sigma_0^+(2, 2, 2) = 8$) we have

$$\sigma_n^\pm(2, 2, 2) = 2\sigma_n^\pm(2, 2) + b_{n-1}(2, 2) + \sigma_{n-2}^\mp(2, 2, 2) - 1 \geqslant n + 1$$

for $n \geqslant 2$.

Next

$$\sigma_n^\pm(2, 3) = \sigma_n^\pm(2, 2) + \sigma_n^\pm(2) + b_{n-1}(2, 2) + \sigma_{n-2}^\mp(2, 2, 3) - 1$$

$$\geqslant b_{n-1}(2, 2) + \sigma_{n-2}^\mp(2, 2, 2) - 1 \quad \text{by (5.4)}$$

$$\geqslant 2n \geqslant 8 \quad \text{for } n \geqslant 4,$$

and $\sigma_2^+(2, 3) = 3$, $\sigma_2^-(2, 3) = 19$.

For the hypersurface of degree 3 we have

$$\sigma_n^\pm(3) = \sigma_n^\pm(2) + \sigma_n^\pm(1) + b_{n-1}(2) + \sigma^\mp(2, 3) - 1 \geqslant 7 \quad \text{for } n \geqslant 6,$$

by the result for $(2, 3)$, and $\sigma_2^+(3) = 1$, $\sigma_2^-(3) = 6$; $\sigma_4^+(3) = 21$, $\sigma_4^-(3) = 2$. Finally one can check $\sigma_2^+(4) = 3$, $\sigma_4^-(4) = 42$, and $\sigma_2^-(2, 2, 2) = 15$. These facts together with Lemma 5.4 complete the proofs of Theorems 5.1 and 5.3.

6. Cases where $h^\perp$ is definite. As a consequence of §5 we have

COROLLARY 6.1. *The only complete intersections of dimension $\geqslant 2$ for which the intersection form on the subgroup $h^\perp$ of vanishing cycles is definite are CP_n, $X_n(2)$, $X_n(2, 2)$, and $X_2(3)$.*

Deligne has given a precise description of the intersection form on $h^\perp$ in these cases [4, p. 339]. The elements of $h^\perp$ of norm $2(-1)^{n/2}$, that is the algebraic vanishing cycles, form a system of roots, R. The Weyl group, which can be identified with a monodromy group, acts transitively on R. It follows that R is of type A, D, or E.

The quotient group $(h^\perp)^\#/h^\perp$ is cyclic of order $h \cdot h$ (3.6). Finally the rank of $h^\perp$ was computed in §5. These facts suffice to determine R (see [1, Chapitre 6]).

PROPOSITION 6.2 (DELIGNE). *For $X_n(2)$, $X_2(3)$, and $X_n(2, 2)$, the intersection form on $h^\perp$ is associated with the Dynkin diagram of type A_1, E_6, and D_{n+3}, respectively.*

It follows that $h^\perp$ has no unimodular summand. For if $h^\perp$ were written as a (not necessarily orthogonal) direct sum, since each root has norm $2(-1)^{n/2}$ and $h^\perp$ is definite of even type, each root must lie in one of the summands. But the Weyl group acts irreducibly, hence one of the summands is zero [1, V.1.2.5]. (Hence the hypothesis on the signature in Theorem A is necessary.) Consequently these spaces have no connected sum decomposition.

Also we have an interesting example of the necessity of the condition rank $H - 4 \geqslant |\text{sign } H|$ in Wall's theorem [24, p. 337]. For $n \equiv 2 \mod 4$, $H = H_n(X_n(2, 2))$ is indefinite. It has odd type if $n \equiv 2 \mod 8$ and even type if $n \equiv 6 \mod 8$. In either case consider the based inner product space (H', h') of §3 which has the same rank,

signature, type, and degree as (H, h). Since H is indefinite H and H' are isomorphic but since $h^\perp = D_{n+3}$ is irreducible while $h'^\perp$ is reducible there is no isomorphism of H to H' carrying h to h'.

The information in 6.2 and in §5 suffices to determine the homology of these varieties as a based inner product space. Consider the intersection of two quadrics.

Let (H, h) be any based inner product space with $h \cdot h = 4$ and H definite or nearly definite (rank $H - |\text{Sign } H| = 0$ or 2). By 3.4, if

$$\text{rank } H \equiv 0 \text{ mod } 8, \text{ then } H \text{ is definite of even type}$$

2	indefinite	even
4	definite	odd
6	indefinite	odd

We will determine the structure of (H, h) with the additional hypothesis that $h^\perp$ is given by the Dynkin diagram of type D. We first collect some properties of a particular example.

Let $e_1, \ldots, e_n$ be the standard orthonormal basis of Q^n, let $n \equiv 0 \text{ mod } 4$, and let Γ_n be the unimodular sublattice of Q^n generated by $e_i + e_j$ and $\frac{1}{2}(e_1 + \cdots + e_n)$ (see [19, p. 27] or [22, p. 87]).

Then $\Sigma a_i e_i \in \Gamma_n$ iff $2a_i \in Z$, $a_1 \equiv a_2 \equiv \cdots \equiv a_n \text{ mod } 1$, and $\Sigma a_i \in 2Z$. Let $h = 2e_1$ and $y = \frac{1}{2}(e_1 + \cdots + e_n)$. Then $h, y \in \Gamma_n$ and $h \cdot h = 4$, $h \cdot y = 1$, $y \cdot y = n/4$. The elements $y, e_2 + e_3, -e_2 + e_3, e_3 - e_4, -e_4 + e_5, \ldots, e_{n-1} - e_n$ form a basis for Γ_n. It is fairly easy to check that the generators $e_i + e_j$ are in the span of this set of elements so they generate Γ_n, but there are exactly n of them so they give a basis. The last $n - 1$ elements lie in $h^\perp$ and, since $h \cdot y = 1$, it follows they give a basis for $h^\perp$. The corresponding diagram is
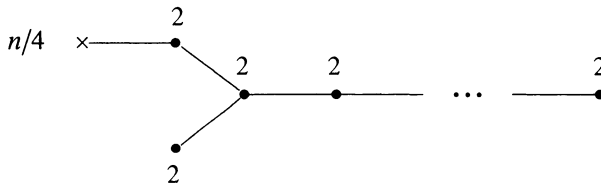
Hence $h^\perp = D_{n-1}$, the bilinear form corresponding to the Dynkin diagram D_{n-1}. Let $x_0 = 4y - h = 2e_2 + \cdots + 2e_n \in D_{n-1}$. Note
(i) x_0 is not divisible by 2 (since $n - 1$ is odd) and
(ii) $x_0 \cdot D_{n-1} \equiv 0 \text{ mod } 4$.
Now the bilinear form space $\langle 4 \rangle \oplus D_{n-1}$ is a sublattice of $Q \oplus Q^{n-1} = Q^n$. Let h denote the generator of $\langle 4 \rangle$ and let $x_0 \in D_{n-1}$ be the element specified above. Then the sublattice of Q^n generated by $\langle 4 \rangle \oplus D_{n-1}$ and the element $\frac{1}{4}(h + x_0)$ is isomorphic to Γ_n with h corresponding to $2e_1 \in \Gamma_n$. Also reflection in the e_1-axis of Q^n induces an isometry of the based inner product space (Γ_n, h) sending x_0 to $-x_0$.

PROPOSITION 6.3. *Let H be a positive definite inner product space of rank $n \equiv 0 \text{ mod } 4$ with an element h such that $h^\perp = D_{n-1}$. Then $(H, h) = (\Gamma_n, 2e_1)$.*

We thank Gunter Harder for conversations providing the proof. The orghogonal sum $hZ \oplus D_{n-1}$ is contained in H as a sublattice of index 4. In fact

$$0 \to hZ \oplus D_{n-1} \to H \to Z/4Z \to 0$$

is exact where $u \in H$ is mapped to $u \cdot h \mod 4$. Since H is unimodular there is a class $y \in H$ with $y \cdot h = 1$. Then $x = 4y - h \in h^{\perp} = D_{n-1}$ and $H = \frac{1}{4}(h + x)Z + (hZ \oplus D_{n-1})$. Since $y \cdot x \equiv 1 \mod 2$, (i) x is not divisible by 2. Also since the form on H is integer-valued, $\frac{1}{4}(h + x) \cdot D_{n-1} \in Z$, so (ii) $x \cdot D_{n-1} \equiv 0 \mod 4$.

If $x_0 \in D_{n-1}$ is another element satisfying (i) and (ii) then $\frac{1}{4}x$ and $\frac{1}{4}x_0$ both generate $D_{n-1}^{\#}/D_{n-1} = Z/4$. Hence $\frac{1}{4}x \equiv \pm \frac{1}{4}x_0 \mod D_{n-1}$ so $x = \pm x_0 + 4u$ for some $u \in D_{n-1}$. But then

$$\tfrac{1}{4}(h + x)Z + (hZ \oplus D_{n-1}) = \tfrac{1}{4}(h \pm x_0)Z + (hZ \oplus D_{n-1}).$$

In either case from the properties of $(\Gamma_n, 2e_1)$ above we have $(H, h) = (\Gamma_n, 2e_1)$.

Let H be indefinite with an element h such that $h^{\perp} = -D_{n-1}$. Then as above $hZ \oplus (-D_{n-1})$ is a sublattice of H, which together with an element $\frac{1}{4}(h + x)$ generates H. The element $x \in -D_{n-1}$ satisfies (i) x is not divisible by 2 and (ii) $x \cdot (-D_{n-1}) \equiv 0 \mod 4$ and x is characterized up to sign modulo $-D_{n-1}$ by these properties. Let $x = 2e_2 + \cdots + 2e_n$ as above so $x \cdot x = -4(n - 1)$. Let $y = \frac{1}{4}(h + x)$. Then $y \cdot (e_2 + e_3) = -1$ and $y \cdot y = \frac{1}{16}(4 - 4(n - 1)) = (2 - n)/4$. Hence, using the same basis for D_{n-1} as above, the intersection diagram for (H, h) is
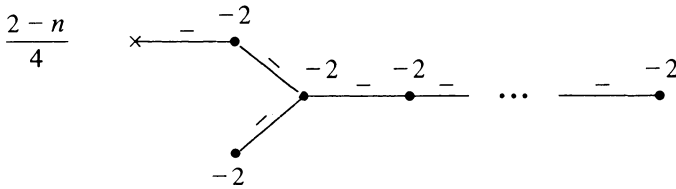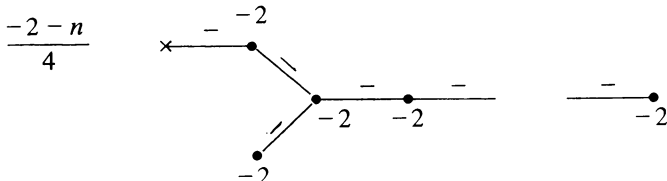

Since the rank of $H_n(X_n(2, 2))$ is $n + 4$, we replace n by $n + 4$ and restate our results.

PROPOSITION 6.4. *Let $H = H_n(X_n(2, 2))$. Then* rank $H = n + 4$. *For $n \equiv 0 \mod 4$, $(H, h) = (\Gamma_{n+4}, 2e_1)$. The intersection diagram is*
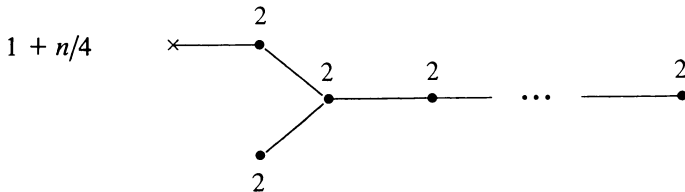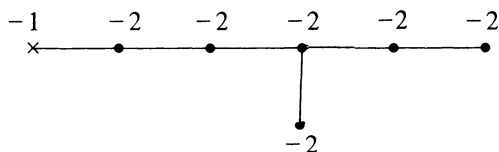

H has odd type if $n \equiv 0 \mod 8$ and even type if $n \equiv 4 \mod 8$. For $n \equiv 2 \mod 4$, the intersection diagram is

In this case sign $H = -n - 2$ *and H has odd type if $n \equiv 2$* mod 8 *and even type if* $n \equiv 6$ mod 8.

REMARK 6.5. Although $X_2(3)$ is of too low dimension for our geometric construc-
tions, for completeness we consider its homology. Recall the intersection form on
$H_2(X_2(3))$ is isometric to $\langle 1 \rangle \oplus 6\langle -1 \rangle$. It follows from the classification of Del
Pezzo surfaces [15, p. 119] that $X_2(3) = CP_2 \# 6\overline{CP}_2$ and, identifying the hyperplane
section, that $h = (3, 1, 1, 1, 1, 1, 1)$.

In fact for this h in $\langle 1 \rangle \oplus 6\langle -1 \rangle$ it is easy to identify $h^\perp$ with E_6; the basis
$y = (0, \ldots, 0, 1)$, $e_1 = (0, 1, -1, 0, 0, 0, 0)$, $e_2 = (0, 0, 1, -1, 0, 0, 0)$, $e_3 =$
$(0, 0, 0, 1, -1, 0, 0)$, $e_4 = (0, 0, 0, 0, 1, -1, 0)$, $e_5 = (0, 0, 0, 0, 0, 1, -1)$, $e_6 =$
$(1, 1, 1, 1, 0, 0, 0)$ has intersection diagram

$$
\begin{array}{cccccc}
-1 & -2 & -2 & -2 & -2 & -2 \\
\times \!\!-\!\!\!-\!\!\!-\!\! \bullet \!\!-\!\!\!-\!\!\!-\!\! \bullet \!\!-\!\!\!-\!\!\!-\!\! \bullet \!\!-\!\!\!-\!\!\!-\!\! \bullet \!\!-\!\!\!-\!\!\!-\!\! \bullet \\
& & & | \\
& & & \bullet \\
& & & -2
\end{array}
$$

The method of 6.3 then implies that any pair (H, h) with $h^\perp = E_6$ is isomorphic to
this example.

7. **Gluing.** The manifolds described by plumbing in §1 are embedded in X as
neighborhoods which retract by deformation onto certain n-dimensional com-
plexes. In this section we give homological conditions on a pair of subcomplexes of
an ambient manifold X which will yield the decomposition of X as in Theorem B.

PROPOSITION 7.1. *Let K_1 and K_2 be CW-complexes of dimension $< n$ disjointly
embedded in X^{2n} such that*:
(i) $H_q(K_j) \to H_q(X)$ *is iso for $q < n$ and mono for $q = n$.*
(ii) $H_n(K_1)$ *is a summand of $H_n(X)$.*
(iii) $H_n(K_2) = H_n(K_1)^\perp$ *in $H_n(X)$,*
(iv) $H^n(K_1) = \mathrm{Hom}(H_n(K_1), Z)$.
Then $H_*(K_2) \to H_*(X - K_1)$ is an isomorphism.

We postpone the proof to the end of this section. Now let W_1 and W_2 be closed
manifolds which are neighborhoods of K_1 and K_2 such that $K_j \hookrightarrow W_j$ is a homotopy
equivalence. Then

$$H_*(X - \mathring{W}_1 \cup \mathring{W}_2, \partial W_2) = H_*(X - \mathring{W}_1, \mathring{W}_2) \quad \text{by excision}$$

$$= H_*(X - K_1, K_2) \quad \text{by homotopy}$$

$$= 0 \quad \text{by exactness.}$$

If X is simply connected, so is $X - \mathring{W}_1 \cup \mathring{W}_2 \simeq X - K_1 \cup K_2$ ($\simeq$ denotes homo-
topy equivalence) since a 2-disk with boundary in $X - K_1 \cup K_2$ can be pushed off
$K_1 \cup K_2$ by general position (n is > 2). If we also assume that ∂W_1 and ∂W_2 are

simply connected, then by the h-cobordism theorem [18, p. 107], $X - \mathring{W}_1 \cup \mathring{W}_2 = \partial W_2 \times I$ ($=$ denotes diffeomorphism). Since $W_2 = W_2 \cup (\partial W_2 \times I)$ we have proved

COROLLARY 7.2. *Under these hypotheses* $X = W_1 \cup_\varphi W_2$ *where* $\varphi\colon \partial W_1 \to \partial W_2$ *is a diffeomorphism.*

To produce K_1 and K_2 we study $H_n X$ and construct a basis whose intersection matrix corresponds to the plumbing diagram for W_1 and W_2.

PROOF OF 7.1. Since K_1 is n-dimensional and by (ii) and (iv) $H^n(X) \to H^n(K_1)$ is onto, the horizontal maps

$$\begin{array}{ccc} H^{2n-q}(X, K_1) & \to & H^{2n-q}(X) \\ \downarrow = & & \downarrow = \\ H_q(X - K_1) & \to & H_q(X) \end{array}$$

are isomorphisms for $q < n$.

Also by (i) $H_q(X - K_1) = H^{2n-q}(X, K_1) = 0$ for $q > n$ and $H_n(X - K_1) \to H_n(X)$ is injective. It remains to show $H_n(K_2) \to H_n(X - K_1)$, which follows from

LEMMA 7.3. $\mathrm{im}\{H_n(X - K_1) \to H_n(X)\} = H_n(K_1)^\perp$ *in* $H_n(X)$.

PROOF. Let $i\colon K_1 \hookrightarrow X$. Then $u \in \mathrm{im}\{H_n(X - K_1) \to H_n(X)\}$ iff $i^* Du = 0$ where $Du \in H^n(X)$ is Poincaré dual to u.

$$\begin{array}{ccccc} H_n(X - K_1) & \to & H_n(X) & & \\ \uparrow = & & D\downarrow = & & \\ H^n(X, K_1) & \to & H^n(X) & \overset{i^*}{\to} & H^n(K_1) \end{array}$$

But $i^* Du = 0$ iff $i^* Du \cap \mu = 0$ for all $\mu \in H_n(K_1)$, by (iv), iff $0 = i_*(i^* Du \cap \mu) = Du \cap i_* \mu = u \cdot i_* \mu$. So

$$\mathrm{im}\{H_n(X - K_1) \to H_n(X)\} = \left\{u \in H_n X\colon u \cdot i_* \mu = 0 \text{ for all } \mu \in H_n(K_1)\right\}$$

$$= \left(i_* H_n(K_1)\right)^\perp \subset H_n(X).$$

This proves 7.3 and completes the proof of 7.1.

8. Even type. In this section we prove Theorem B in the case where H has even type. Recall s is the number of even entries in the multidegree **d**.

THEOREM 8.1. *If* $n = 2m > 2$, $\binom{m+s}{m}$ *is even and* $\mathbf{d} \neq (2, 2)$, *then* $X_n(\mathbf{d}) = W_1 \cup_\varphi W_2$ *where* $\varphi\colon \partial W_1 \to \partial W_2$ *is a diffeomorphism,*

$$W_2 = W \natural \alpha(S^n \times S^n - D^{2n}) \natural \beta V,$$

and where W *and* W_1 *are two copies of the* n-*disk bundle over* CP_m *with Euler class zero and stable class given by 2.3.*

PROOF. $X_n(2, 2)$ does not admit such a decomposition by §6. In all other cases where H has even type, by 5.1 and 3.2(a) we have $H = A \oplus B$ where rank $A = 2$

and $B = \alpha U \oplus \beta \Gamma_8$. A basis for B with the given intersection matrices can be represented by embedded copies of S^n, by 2.1, and by the Whitney process we may suppose the geometric intersections of these spheres are exactly the algebraic intersections. Thus each U summand is represented by two copies of S^n embedded with exactly one point of transverse intersection. The normal bundles are trivial since they are stably trivial and have Euler class zero. Therefore a neighborhood is diffeomorphic to the manifold $S^n \times S^n - D^{2n}$ obtained by plumbing according to the diagram

$$\underset{\bullet}{0} \underset{\,}{—} \underset{\bullet}{0} .$$

In the same way the Γ_8 summands are represented by disjointly embedded copies of the manifold V obtained by plumbing according to the diagram E_8. (It follows that $X = M \# \alpha(S^n \times S^n) \# \beta(V \cup D^{2n})$ which gives Corollary C except for the decomposition result for M.)

Now referring to the proof of 3.2(a) there is a class $e \in A$ with $e \cdot h = 1$ and $e \cdot e = 0$. The case of the quadric $X_{2m}(2)$, which has even type when m is odd, is not covered by 3.2(a) because rank $H = 2$. But then $H = U$ with basis e, f. Since $h \cdot h = 2$ we must have $h = e + f$ (or $-e - f$, but in that case change the basis). Then the class $e \in H$ has the desired properties. Such a class does not exist in $H_n(X_n(2, 2))$ since then h, e would have intersection matrix $\begin{pmatrix} d & 1 \\ 1 & 0 \end{pmatrix}$ which is unimodular, contradiciting the remarks following 6.2.

By 2.1 e is represented by an embedded CP_m in X. Denote the image of CP_m by K and let W be the normal disk bundle. Since the Euler class of this normal bundle is zero, there is a homologous, disjoint copy which we denote by K_1 with normal bundle W_1. Finally let $W_2 = W \natural \alpha(S^n \times S^n - D^{2n}) \natural \beta V$ where the boundary connected sum is formed inside X by sending out thin pseudopods. W_2 retracts by deformation onto an n-complex K_2 obtained by connecting by arcs the complexes V, the α copies of $S^n \vee S^n$, and the $|\beta|$ copies of the core of V. These spaces satisfy the hypotheses of 7.1 and 7.2 and this proves 8.1.

We refer to the construction used here as plumbing inside a manifold.

9. Plumbing. In this section we complete the proof of Theorem B. We also prove that the manifold W in the decomposition is independent of various choices made in its construction and give another description of W based on Haefliger's work on differentiable links.

THEOREM 9.1. *If $n = 2m > 2$, then $X_n(\mathbf{d}) = W_1 \cup_\varphi W_2$ where W_1 is an $(n + 2)$-disk bundle over CP_{m-1} and $W_2 = W \natural \alpha(S^n \times S^n - D^{2n}) \natural \beta V$ is obtained by plumbing inside X. If $\mathbf{d} \neq (2, 2)$, rank $H_n W \leqslant 5$.*

REMARK. With this construction all of H is carried by W_2.

PROOF. By 3.2 and 5.1 we have $H = A \oplus B$ with $h \in A$ and with rank $A \leqslant 5$ unless $\mathbf{d} = (2, 2)$. In those other cases where 3.1 does not apply we may take $A = H$. Since h is indivisible and A is unimodular there is a class $y \in A$ with $y \cdot h = 1$. Then for any $x \in A$, $x - (h \cdot y)y \in h^\perp$ so $A = \mathbf{Z}y + h^\perp$. Choose a basis $y, e_1, \ldots, e_k$ for A with $e_i \in h^\perp$. Then by 2.1 y is represented by an embedded

$CP_m \subset X$ and each e_i by an embedded $S^n \subset X$. By the Whitney process we may suppose the geometric intersection of these manifolds is given by the inner product on A.

Then inside X we have a $2n$-manifold with boundary P obtained by plumbing a bundle over CP_m with bundles over S^n. However P will not necessarily be simply connected (cf. [2, p. 119]). Choose loops embedded in ∂P representing generators of the free group $\pi_1(\partial P) = \pi_1(P)$. These bound 2-disks properly embedded in $\overline{X - P}$, so $(D^2, \partial D^2) \subset (\overline{X - P}, \partial P)$ since int D^2 can be pushed off the core of P by geneal position. The normal bundle to each D^2 in X is trivial so we may attach $D^2 \times D^{2n-2}$ to P along $S^1 \times D^{2n-2} \subset \partial P$ to obtain a simply connected $2n$-manifold with boundary W. Let K be obtained from the union of CP_m and the n-spheres by attaching 2-disks which extend the 2-disks attached above to ∂P. Then K is an n-complex contained in W as a deformation retract. K carries the summand A of the homology of X.

As in §8 the summand B is represented by $\alpha(S^n \times S^n - D^{2n}) \natural \beta V \subset X$ and we take $W_2 = W \natural \alpha(S^n \times S^n - D^{2n}) \natural \beta V$ and construct an n-complex K_2 which is a deformation retract of W_2.

Since $i: X_n \to CP_{n+r}$ is an n-equivalence there is a map $j: CP_{m-1} \to X_n$ unique up to homotopy such that $i \circ j$ is the linear inclusion. We may suppose j is an embedding disjoint from W_2. Let K_1 be the image of j and W_1 be the normal $(n + 2)$-disk bundle. This is a stable bundle determined by the equation $\tau CP_{m-1} \oplus \nu(j) = \tau X$ as in 2.3.

Then 7.1 applies to K_1 and K_2 in X and by 7.2 $X = W_1 \cup_\varphi W_2$. This completes the proof of 9.1.

There are some consequences of the proof. If we are given a decomposition $H = A \oplus B$ and a basis $y, e_1, \ldots, e_k$ for A then the intersection diagram corresponding to this basis is the plumbing diagram for W. In particular if the diagram is simply connected, then we take $W = P$; the construction is described explicitly by the diagram without additional handles to make W simply connected.

COROLLARY 9.2. *If $H = H_n X_n(\mathbf{d})$ has odd type and degree d with $n \geqslant 4$, $\sigma^+(H) \geqslant d$, and $\mathrm{rank}\, H - |\mathrm{Sign}\, H| \geqslant 4$, then there is a decomposition of X as in 9.1 with W constructed according to the diagram*
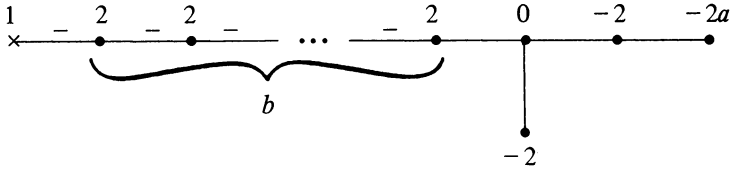

$$d - 1$$
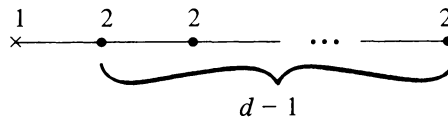
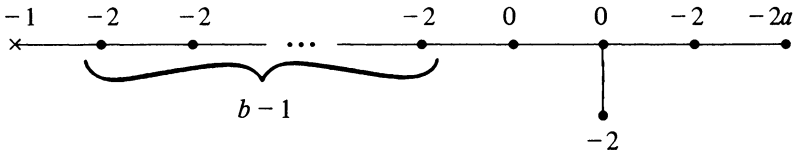Applying these techniques to the decomposition of H given in 4.1 we obtain

PROPOSITION 9.3. *If $n = 2m \geqslant 4$ and $X_n(\mathbf{d})$ has odd type with $\mathbf{d} \neq (2)$ or $(2, 2)$, and for $n = 4$, $\mathbf{d} \neq (3)$, then $X = W_1 \cup_\varphi W_2$ where W_1 is an n-disk bundle over CP_m with Euler class 1 and stable class given by 2.3, and*

$$W_2 = W \natural \sigma(S^n \times S^n - D^{2n}) \natural \beta V$$

where W is constructed by plumbing according to the simply connected diagram

for d = 8a + b, 0 < b ⩽ 4, or

for d = 8a − b, 0 < b ⩽ 4.

If we are given an intersection diagram, let the vertices correspond to basis elements $y, e_1, \ldots, e_k$ of a free Z-module A and define a bilinear form in A using the matrix corresponding to the diagram. If A is unimodular a class $h \in A$ is uniquely determined by the equations $h \cdot y = 1$, $h \cdot e_j = 0$. If each $e_j \cdot e_j$ is even, $h^\perp$ has even type and (A, h) is a based inner product space.

LEMMA 9.4. *Let (A, h) be a based inner product space with basis $y, e_1, \ldots, e_k$ such that $h \cdot y = 1$ and $h \cdot e_j = 0$. Let η be a stable bundle over CP_m with $w_{2m}(\eta) = (type of A) \in Z/2$. For $n = 2m \geqslant 4$, there is a simply connected 2n-manifold with boundary W constructed by plumbing with $H_n W = A$. If $h \in H_n W$ corresponds to $h \in A$, then if $u \cdot h = 0$, u is represented by an embedded $S^n \subset \text{int } W$ and if $u \cdot h = 1$, u is represented by an embedded CP_m.*

PROOF. This follows from [2, V.2.1]. Since $h^\perp$ has even type, there are stably trivial n-plane bundles over S^n with Euler class $e_j \cdot e_j$. Since $y \cdot y$ mod 2 is the type of A, there is an n-plane bundle over CP_m representing η with Euler class $y \cdot y$ (cf. the proof of 2.4). Plumbing these bundles together according to the intersection diagram of A with respect to the given basis we obtain a manifold with boundary P. If the diagram is not simply connected we attach handles $D^2 \times D^{2n-2}$ to obtain a simply connected W. Given u with $u \cdot h = 0$, $u \in \text{span}\{e_1, \ldots, e_k\} \subset A$ and hence $u \in \text{im}\{\pi_n W \to H_n W\}$. By [6] u is represented by an embedded sphere. The case $u \cdot h = 1$ follows also as in 2.1.

COROLLARY 9.5. *W is determined up to diffeomorphism by (A, h) and η.*

PROOF. Given another basis $y', e_1', \ldots, e_u'$ for A construct W' by plumbing inside W. Then $W' \to W$ is a homotopy equivalence and $\overline{W - W'}$ is an h-cobordism so W is diffeomorphic to W'.

We conclude this section with a description of W in terms of surgery on a link. Let E be the given n-disk bundle over CP_m. Let D^n be a small disk in CP_m and let

$S^{2n-1} = \partial(E|D^n)$. A link of $(n-1)$-spheres in S^{2n-1} is in the stable range [7, §5] and is determined up to isotopy by the linking elements $\lambda_j^i \in Z$ for $i < j$. There is a framed link of spheres $S_0^{n-1}, S_1^{n-1}, \ldots, S_k^{n-1}$ in S^{2n-1} such that

(i) $S_0^{n-1} = CP_m \cap S^{2n-1} = \partial D^n$,

(ii) $\lambda_j^i = e_i \cdot e_j$ (where $e_0 = y$),

(iii) $e_i \cdot e_i \in Z = \ker\{\pi_{n-1}(SO(n)) \to \pi_{n-1}(SO)\}$ determines the normal framing.

Then attach handles $D^n \times D^n$ along $S^{n-1} \times D^n$ to the framed spheres $S_1^{n-1}, \ldots, S_k^{n-1}$ to obtain a $2n$-manifold with boundary W'. doing plumbing inside W' as above we conclude that W' is diffeomorphic to the manifold W.

10. Examples. According to Corollary C and the results of §9 the manifold M which remains after removing a $(2m-1)$-connected $4m$-manifold with maximal Betti number from a complete intersection can be represented as the gluing of a linear disk bundle over CP_{m-1} and a manifold obtained by plumbing bundles over CP_m and copies of S^{2m}. In this section we compute the diagrams describing such plumbings for complete intersections of low degree. (The results of 3.2(a) and §4 give an explicit diagram for an M in all degrees but not always an M with minimal Betti number.) The problem is purely algebraic, namely for a based inner product space (A, h) we seek the intersection matrix for a basis of the special form $y, e_1, \ldots, e_k$ where $y \cdot h = 1$ and $y \cdot e_j = 0$.

EXAMPLE $d = 2$. In this case $A = H$ is the minimal space containing h. If m is even, H has odd type of 2.2 and sign $H = 2$ (by 3.4). Therefore H is equivalent to $\langle 1 \rangle \oplus \langle 1 \rangle$. Since h has norm 2 we may choose the basis so $h = (1, 1)$. Let $y = (1, 0)$ and $e = (1, -1)$. Then the intersection matrix corresponds to the diagram

<div align="center">

1   2

×———————●

</div>
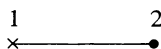
When m is odd, H has even type and is equivalent to the hyperbolic plane U. Choosing the basis so $h = (1, 1)$ and taking $y = (1, 0)$, $e = (-1, 1)$ gives the diagram

<div align="center">

0   -2

×———————●

</div>

An alternate description of $X_n(2)$ in this case, $n = 2m$, m odd, is given by 3.2(a). The algebraically embedded $CP_m \subset X_n(2)$ has self-intersection zero and so can be moved to produce a second disjoint copy. Let the normal disk bundle of CP_m be E. Then $X_n(2) = E \cup_\varphi E$ by §8.

EXAMPLE $d = 10$. Whenever $d \equiv 2 \bmod 8$, the number s of even entries in $\mathbf{d}$ is 1, so the type is odd if m is even. For $d = 10$, by 3.2(b) we may take $A = \langle 1 \rangle \oplus \langle 1 \rangle$. Then $h = (3, 1), y = (0, 1), e = (1, -3)$ gives the diagram

<div align="center">

1   -3 10

×———————●

</div>
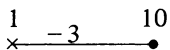
It is easy to show that there is no simply connected diagram of rank 2 for $d = 10$ when H has odd type. A simply connected diagram for $d = 10$ and H of odd type is
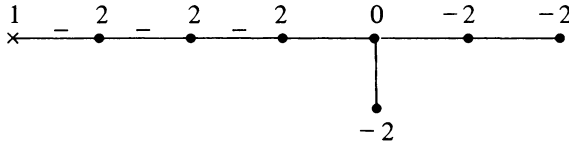
The hypersurface $\mathbf{d} = (10)$ and the complete intersection $\mathbf{d} = (5, 2)$ have the same plumbing diagram. They are distinguished by the stable type of the bundles over CP_m and over CP_{m-1}. The bundle ν over CP_m is determined stably by the equation $\nu + \gamma^{10} = (m + 1)\gamma$ in case $\mathbf{d} = (10)$ and by $\nu + \gamma^5 + \gamma^2 = (m + 2)\gamma$ in case $\mathbf{d} = (5, 2)$.

EXAMPLE $d = 3$. The minimal submodule A is equivalent to $\langle 1 \rangle \oplus \langle 1 \rangle \oplus \langle 1 \rangle$ with $h = (1, 1, 1)$. If $y = (1, 0, 0)$, $e_1 = (1, -1, 0)$, and $e_2(0, -1, 1)$, the diagram is
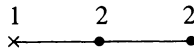

This is a special case of 4.2. Although 4.3 does not apply to $X_4(3)$, this simply connected example does. The case of $X_2(3)$ which has irreducible H is given at the end of §6.

EXAMPLE $d = 4$. By 3.2 rank 4 is minimal; Example 4.2 gives a simple connected diagram of rank 4.
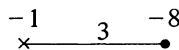
EXAMPLE $d = 5$. By 3.2 rank 5 is minimal. In this case there are two possible signatures. A positive definite example is given by 4.2: $A = 5\langle 1 \rangle$, $h = (1, 1, 1, 1, 1)$, $e_1 = (1, -1, 0, 0, 0)$, $e_2 = (0, 1, -1, 0, 0)$, ..., $y = (1, 0, 0, 0, 0)$. The diagram is
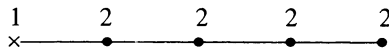
Another possibility is $A = \langle 1 \rangle \oplus 4\langle -1 \rangle$, $h = (3, 1, 1, 1)$, $e_1 = (0, 1, -1, 0, 0)$, $e_2 = (0, 0, 1, -1, 0)$, $e_3 = (0, 0, 0, 1, -1)$, $e_4 = (1, 1, 1, 1, 0)$, $y = (0, 0, 0, 0, 1)$. The diagram is
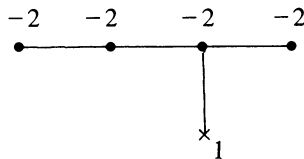
EXAMPLE $d = 8$. We may take $A = \langle 1 \rangle \oplus \langle -1 \rangle$, $h = (3, 1)$, $y = (0, 1)$, and $e = (1, 3)$. The diagram is

In any particular case explicit description of the bundles and the plumbing diagram is straightforward (modulo the sums of squares expressions of the proof of 3.2). For some purposes the explicit, simply connected diagrams of higher rank may be perferable. The outstanding problem is to describe the diffeomorphism φ.

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