Proceedings of Symposia in Pure Mathematics
Volume 40 (1983), Part 2

ALEXANDER INVARIANTS OF PLANE ALGEBRAIC CURVES

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1. Introduction. The aim of this paper is to describe our work [**7, 8, 9**] on the fundamental groups of the complement to irreducible plane algebraic curves. The approach is based on the methods used so far for the study of knot groups. We associate with such a curve an invariant of the fundamental group of the complement, which we call the Alexander module of the curve, and which is essentially equivalent to the factor of this fundamental group by its second commutator subgroup. We show how it depends on the types of singularities of the curve and compute the "rationalized" Alexander module in terms of the position of singularities. The key tool is the study of the surfaces associated with any curve C, namely the cyclic coverings of the projective plane with branching set C and possibly the line L in infinity.

The study of the plane singular curves was initiated by O. Zariski in the beautiful series of papers [**17–22**] which appeared between 1929 and 1936. Most of the ideas described below are taken from those works. In particular Zariski observed and exploited the connection between nontriviality of the first Betti number of the cyclic branched covering and noncommutativity of the fundamental group of the complement to the branching locus. Our Alexander modules give the quantitative measure of this relationship.

2. Experimental data: examples. Before explaining our main results on the fundamental groups of the complement to the plane curves we shall describe known examples of computations of those groups. From now on we shall refer to the fundamental groups of the complement to the plane curve as the groups of the curve (cf. the terminology: the knot group is the fundamental group of the complement to the knotted sphere). Note that constructing computable examples is not an easy task.

1980 *Mathematics Subject Classification.* Primary 14F20, 57M05, 14H20, 57M10, 14J17, 57M15.

[1]Supported by an NSF Grant.

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0082-0717/81/0000-C192/$03.25

EXAMPLE 2.1. Let C be a nonsingular algebraic curve of degree d in P^2. Then $\pi_1(P^2 - C) = Z/d$ is the cyclic group of order d. If C has as singular points only nodes (i.e. all singularities locally can be given by equation $x^2 - y^2 = 0$) then $\pi_1(P^2 - C) = Z/d$ as well. This was known as Zariski's conjecture and proved by Deligne and Fulton in 1979 [4, 6].

EXAMPLE 2.2. Let V be a nonsingular algebraic surface and φ: $V \to P^2$ be a generic projection. Let $C \subset P^2$ be the branching locus of φ. Then C has nodes and cusps as singularities (the latter locally are given by equation $x^2 = y^3$), and C usually has an interesting fundamental group. The concrete computations were made by B. Moishezon [12] in the case where V is a hypersurface in P^3. Let n be the degree of the hypersurface V. Then C is the curve of degree $n(n-1)$ with $n(n-1)(n-2)(n-3)/2$ nodes and $n(n-1)(n-2)$ cusps; $\pi_1(P^2 - C)$ is the Artin's braid group $B(n)$ (see [1]), factorized by the center. In the case $n = 3$ this was computed already by Zariski [17]: one obtains the famous six-cuspidal sextic with all cusps belonging to a conic, and $\pi_1(P^2 - C)$ is given in [17] as the free product $Z/2 * Z/3$.

EXAMPLE 2.3. The following construction goes back to Zariski (cf. [20] and also [5]). Let V be a nonsingular projective variety over $\mathbf{C}$, L an invertible sheaf on V, and E a subspace of $H^0(V, L)$. Then $P(E)$, the projective space associated to E, can be thought as the set of divisors D_x $(x \in P(E))$ on V belonging to the linear system E. Let Disc(E) be the subset of $P(E)$ such that D_x is singular.

In many cases Disc(E) is the singular hypersurface in $P(E)$ with nontrivial $\pi_1(P(E) - \text{Disc}(E))$. For generic plane P^2 in $P(E)$, Disc$(E) \cap P^2$ is the curve C for which by Zariski's theorem [21] $\pi_1(P^2 - C) = \pi_1(P(E) - \text{Disc}(E))$, i.e. one obtains C with a nontrivial fundamental group. The concrete computations were made in the following cases:

(a) (Zariski [20]). Let $V = P^1$, $L_1 = O(n)$ and $E = H^0(P^1, O(n))$. In down to earth terms $P(E)$ can be thought of as the set of unordered n-tuples of points on P^1 (which is parametrized by P^n) and Disc(E) as the set of n-tuples which have coincident points. The curve $C = \text{Disc}(E) \cap P^2$ for generic P^2 has degree $2(n-1)$, $(3n-6)$ cusps and $2(n-2)(n-3)$ nodes (in fact C is dual to a generic rational nodal curve of degree n). $\pi_1(P^2 - C)$ is the braid group of sphere (see [1]), i.e. has a presentation with generators $\sigma_1 \cdots \sigma_{n-1}$ and relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| \geq 2$, $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, and $\sigma_1 \cdots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2} \cdots \sigma_1 = 1$.

(b) (cf. [5]). Let $V = P^2$ (resp. $V = P^1 \times P^1$), $L = O_{P^2}(3)$, (resp. $L = O_{P^1}(2) \otimes O_{P^1}(2)$) and E be the complete linear system defined by L. In this case $P(E) - \text{Disc}(E)$ can be thought of as the space of nonsingular plane cubic curves (resp. nonsingular sections of a fixed quadric in P^3 by other quadrics). The construction above yields the curve C_3 of degree 12 with 24 cusps and 21 nodes (resp. the curve $C_{2,2}$ of degree 12 with 24 cusps and 22 nodes). The fundamental groups of those curves are given as the following semidirect products:

$$0 \to K_{27} \to \pi_1(P^2 - C_3) \to \mathrm{SL}_2(Z) \to 1,$$
$$0 \to Q_{64} \to \pi_1(P^2 - C_{2,2}) \to \mathrm{SL}_2(Z) \to 1$$

where K_{27} (resp. Q_{64}) is the unique nonabelian group of order 27 (resp. 64) and of exponents 3 (resp. 4).

Note that the partial information on $\pi_1(P(H^0(P^3, O_{P^3}(3)) - \text{Disc}(E)))$ (i.e. on the π_1 for the space of nonsingular cubic surfaces) is obtained in [10].

(c) [22]. Let V be an elliptic curve, D be any divisor of degree n, and $E = H^0(V, L(D))$. The curve $C = \text{Disc}(E) \cap P^2$ for generic P^2 is the curve of degree $2n$ with $3n$ cusps and $2n(n-3)$ nodes. It can be identified with the dual to generic plane curve of degree n and genus 1. The fundamental group $\pi_1(P^2 - C)$ can be identified with $H_n = \text{Ker}(B_n(T) \overset{\varphi}{\to} H_1(T))$ where $B_n(T)$ is the braid group of torus T [1], and φ is the natural projection. In [22] Zariski gives an explicit presentation for H_n.

EXAMPLE 2.4 (OKA [13]). Let $O_{p,q}$ be given by equation $(X^p + Y^p)^q + (Y^q + Z^q)^p = 0$ where p and q are relatively prime. This curve has pq singularities given locally by equation $X^p + Y^q = 0$. Oka's computation yields $\pi_1(P^2 - O_{p,q}) = Z_p * Z_q$. (Case $p = 2$, $q = 3$ is again the case considered by Zariski in Example 2.2 $(n = 3)$.)

3. Definition of the Alexander module. Although in the geometric problems the main role is apparently played by fundamental groups of the complement to projective curves, we will study the groups of curves in an affine plane. Moreover for simplicity we shall assume that the line L in infinity is in a general position relative to C. Most of the results below can be modified to include the case of arbitrary position of L (cf. [7]). In the case of generic L the groups of affine and projective curves are related by the central extension (cf. [19])

$$0 \to Z \to \pi_1(P^2 - (C \cup L)) \to \pi_1(P^2 - C) \to 0. \tag{3.1}$$

Therefore essentially no information is lost by switching to affine curves.

Let G denote $\pi_1(P^2 - (C \cup L))$, and G' and G'' be the first and the second commutator subgroups of G. We have the extension

$$0 \to G'/G'' \to G/G'' \to G/G' \to 0. \tag{3.2}$$

One can easily see that

$$G/G' = H_1(P^2 - (C \cup L), Z) = Z. \tag{3.3}$$

Hence to describe G/G'' it is enough to describe G'/G'' as the module over the Z-group ring of Z, i.e. over the ring $Z[t, t^{-1}]$ of Laurent polynomials with integral coefficients.

DEFINITION 3.1. The Alexander module $A_1(Z)$ of a plane algebraic curve is the group G'/G'' considered as a $Z[t, t^{-1}]$-module.

One can give a slightly more geometric description of the Alexander modules as follows (cf. [11]). The group G'/G'' can be identified with $H_1(\widetilde{P^2 - C \cup L})$, the first homology group of the infinite cyclic covering of $P^2 - C \cup L$. The action of $G/G' = Z$ on G'/G'' from the sequence (3.2) coincides with the action of Z on

G'/G'' by the deck transformation. This motivates the following

DEFINITION 3.2. Let R be a ring. The ith Alexander module of the curve C with coefficients in R is the group $H_i(P^2 - C \cup L, R)$ considered as a $R[t, t^{-1}]$ module. If R is a field then the Alexander modules have a more simple structure.

LEMMA 3.1 (CF. [11, 7]). *Assume that R is a field. Then $A_1(R)$ is a $R[t, t^{-1}]$-torsion module. In particular, because $R[t, t^{-1}]$ is a principal ideal domain, we have the cyclic decomposition*

$$A_1(R) = \bigoplus_i R[t, t^{-1}]/(\lambda_i) \tag{3.4}$$

where (λ_i) is the principal ideal generated by the polynomial λ_i.

DEFINITION 3.3. The (global) Alexander polynomial $\Delta_C(R)$ of the curve C is $\Pi_i \lambda_i$.

In the cases when the fundamental group of C is known one can easily compute the corresponding Alexander module and polynomial (cf. [7, 8]). Main results of this paper give information on $A_1(Q)$ and $\Delta_C(R)$ which does not depend on the knowledge of $\pi_1(P^2 - C)$. Here are the computations of Alexander polynomials for examples given in §2.

EXAMPLE 3.1. Let C_n be the branching curve of a generic projection of a nonsingular hypersurface in P^3. Then

$$\Delta_{C_n}(Z) = \begin{cases} t^2 - t + 1, & n = 3, 4, \\ 1, & n \geq 5. \end{cases} \tag{3.5}$$

EXAMPLE 3.2. Let C be a 3-cuspidal quartic (Example 2.3(a), $n = 3$). Then $\pi_1(P^2 - C)$ is the metacyclic group of order 12, and

$$\Delta_C(F_p) = \begin{cases} 1, & p \neq 3, \\ t + 1, & p = 3 \end{cases} \tag{3.6}$$

(F_p is the prime field of characteristic p).

DEFINITION 3.4. Let p_i be a singularity of a curve C. The local Alexander polynomial of p_i is the characteristic polynomial of the monodromy operator of the singularity p_i. Equivalently, the Alexander polynomial of p_i is the Alexander polynomial in the sense of the knot theory of the link of singularity p_i. We denote it by $\Delta_{p_i}(R)$.

EXAMPLE 3.3. The local Alexander polynomial of the node is equal to $t - 1$.

The local Alexander polynomial of the cusp $x^2 = y^3$ is equal to $t^2 - t + 1$. More generally for the singularity $x^p = y^q$ with relatively prime p and q one obtains as the Alexander polynomial

$$\Delta_{p,q} = \frac{(t^{pq} - 1)(t - 1)}{(t^p - 1)(t^q - 1)}.$$

REMARK 3.1. The definition of the local Alexander polynomial requires modification in the case when L is not in general position relative to C. Then one should consider the points of tangency of L and C as singular points (cf. [7]).

DEFINITION 3.5. Let $S^3 = \partial T(L)$ be the boundary of the tubular neighbourhood of L. The Alexander polynomial of C in infinity is the Alexander polynomial of the link $C \cap \partial T(L) \subset \partial T(L)$. We denote it as $\Delta_{\infty,C}(R)$.

EXAMPLE 3.4 (CF. [7]). Let C be in general position relative to L, and let the degree of C equal d. Then

$$\Delta_{\infty,C}(R) = (t - 1)(t^d - 1)^{d-2}.$$

4. Divisibility theorem and applications.

THEOREM 4.1 [7]. *Let R be a field and let C be an irreducible plane algebraic curve. Then*:

(1) $\Delta_C(R)$ *divides the product* $\Pi \Delta_{p_i,c}(R)$ *of the local Alexander polynomials of all singularities.*

(2) $\Delta_C(R)$ *divides* $\Delta_{\infty,C}(R)$.

We shall indicate two applications of this theorem by specializing it to the case of curves which have as singularities only cusps and nodes. The first one is

COROLLARY 4.1 (CF. [8]). *Let C be a cuspidal curve.*

(a) *If* $\deg C \equiv \pm 1$ (6) *then G' is a perfect group.*

(b) *If* $\deg C \equiv \pm 2$ (6) *then G'/G'' has only 3-torsion.*

(c) *If* $\deg C \equiv 3$ (6) *then G'/G'' has only 2-torsion.*

Indeed, the Alexander polynomial $\Delta_C(F)$ should divide $\Pi \Delta_{p_i,C} = (t^2 - t + 1)^\kappa \cdot (t - 1)^\delta$ and $\Delta_{\infty,C} = (t - 1)(t^d - 1)^{d-2}$, where κ is the number of cusps and δ is the number of nodes. If $p = 2$ (resp. $p = 3$), then those two polynomials do not have common roots besides $t = 1$ (i.e., $\Delta_C(F_p) = 1$) unless $\deg C \equiv 0, \pm 2$ (6) (resp. unless $\deg C \equiv 0, 3$ (6)).

For other p's $\Pi \Delta_{p_i,C}$ and $\Delta_{\infty,C}$ do not have common roots besides $t = 1$ if $\deg C \not\equiv 0$ (6). This clearly implies Corollary 4.1.

REMARK 4.1. R. Randell explained to me that this corollary can be deduced from the results of [14] as well.

REMARK 4.2. G'/G'' can indeed have 3-torsion as shown by Example 3.2.

Another application of Theorem 4.1 is the following

COROLLARY 4.2 [18, 9]. *Let $F_k(C)$ be a desingularisation of the k-fold cyclic multiple plane, i.e. the cyclic covering of P^2 branched over C and possibly the line L in infinity. Then the irregularity of $F_k(C)$ is zero unless both k and* $\deg C$ *are divisible by* 6.

The proof of this corollary is based on the following

LEMMA 4.1 (CF. [7]). *Let $F_k(C)$ be as in Corollary 4.2 and let λ_i be polynomials defined by the cyclic decomposition (3.4) of $A_1(Q)$. Let c_i^k be the number of the common roots of λ_i and $t^k - 1$. Then* rk $H_1(F_k(C), Q) = \Sigma_i c_i^k$.

Clearly the divisibility theorem and Lemma 4.1 imply Corollary 4.2. Indeed, if $6 \nmid \deg C$ then $\Delta_C(Q) = 1$. If $6 \mid \deg C$ then $\Delta_C(Q)$ is equal to $(t^2 - t + 1)^l$ for some l. In both cases $\Delta_C(Q)$ and hence all λ_i are relatively prime to $t^k - 1$, i.e. $c_i^k = 0$.

More detailed analysis of $F_k(C)$ leads to the following result.

THEOREM 4.2 (CF. [8]). *Let $F_k(C)$ be as in Corollary 4.2. If all singularities of C are unibranched or nodes and $\exp(2\pi i/k)$ is a root of none of the Alexander polynomials of branches of C, then $F_k(C)$ is simply connected.*

COROLLARY 4.3. *If k is a power of a prime and singularities of C are as in Theorem 4.2 then $F_k(C)$ is simply connected.*

This follows from the standard fact that the Alexander polynomial can be normalized as $\Delta_C(1) = 1$ and the fact that no divisor φ of $t^{p^l} - 1$ has property $\varphi(1) = 1$.

We conclude this section with a result in a different direction. Let $\mathcal{C}$ denote an immersed Riemannian surface in CP^2 with possibly nonlocally flat points. It is interesting to know how the algebraicity of $\mathcal{C}$ affects the topology of the complement of $\mathcal{C}$ in CP^2. One can define in a similar way the Alexander polynomial of $\mathcal{C}$. However the divisibility theorem as above is false for nonalgebraic immersions. In fact, $\Delta_{\mathcal{C}}$ fails to be even cyclotomic. The dichotomy between algebraic and nonalgebraic cases is reflected also in $A_2(Z)$.

THEOREM 4.3 [8]. *Let $\mathcal{C}$ be an immersed surface in CP^2 and $A_i(Z)$ be the ith Alexander module of $\mathcal{C}$ (see Definition 3.2). Then*:

(a) *$A_i(\mathcal{C}) = 0$ for $i > 2$.*

(b) *If $\mathcal{C}$ is algebraic then $A_2(Z)$ is a free $Z[t, t^{-1}]$ module of rank* $\operatorname{rk} H_1(\mathcal{C}) + \deg \mathcal{C} - 1$.

REMARK 4.3. Another interpretation of the global Alexander polynomial was given by R. Randell (see [15]).

5. Computation of $A_1(Q)$. In this section, the Alexander module $A_1(Q)$ will be computed in terms of dimensions of certain linear systems of curves which have "prescribed behavior" at the singular points of the curve C. To describe these linear systems we need to introduce several notions.

PROPOSITION 5.1. *Let $f(x, y)$ be a germ of an analytic function which has an isolated singularity at the origin and let φ be any polynomial. Let $\psi_\varphi(n)$ denote the minimal k such that $z^k\varphi$ belongs to the adjoint ideal* [2] *of the singularity $z^k = f(x, y)$. Then $\psi_\varphi(n)$ is either zero, or has the form $[\kappa_\varphi n]$ where κ_φ is a rational number.* ([ ] *denotes the integral part.*)

The function $\psi_\varphi(n)$ can be computed in terms of an embedded desingularisation of f. In the case when f is generic in the sense of Kouchnirenko one can use the following description of the adjoint ideal given by Tessier and Merle. According to [16] in the case when $F(x, y, z)$ has a nondegenerate isolated singularity the adjoint ideal is generated by the monomials $x^{i_0}y^{i_1}z^{i_2}$ such that $(i_0 + 1, i_1 + 1, i_2 + 1)$ is inside the Newton polyhedron of the singularity $F(x, y, z)$.

DEFINITION 5.1. A constant of quasiadjunction of the singularity $f(x, y)$ is one of the numbers κ_φ for various φ. A constant of quasiadjunction of the curve C is one of the constants of quasiadjunction of the singularities of C.

EXAMPLE 5.1. (a) Let $f(x, y) = x^2 + y^2$. This singularity has no constants of quasiadjunction. (b) Let $f(x, y) = x^2 + y^3$. There is only one constant of quasiadjunction, namely $\kappa_1 = \frac{1}{6}$ corresponding to $\varphi = 1$. (c) Let $f(x, y) = x^2 + y^5$. There are two constants of quasiadjunction: $\kappa_1 = \frac{3}{10}$ and $\kappa_x = \frac{1}{10}$ corresponding to $\varphi = 1$ and $\varphi = x$.

These computations follow from the explicit computation of the Newton polyhedra for corresponding singularities.

DEFINITION 5.2. Let κ be a constant of quasiadjunction of the curve C. Then $\mathcal{Q}_\kappa$ is the sheaf of ideals in $\mathcal{O}_{P^2}$ such that $\Gamma(U, \mathcal{Q}_\kappa)$ consists of $\varphi \in \Gamma(U, \mathcal{O}_{P^2})$ such that $\kappa_\varphi < \kappa$ at any singular point of C belonging to U.

DEFINITION 5.3. The superabundance $s_\kappa(n)$ of the linear system $\Gamma(P^2, \mathcal{Q}_\kappa(n))$ of the curves of degree n defined by sheaf A_κ is $\dim H^1(P^2, \mathcal{Q}_\kappa(n))$.

DEFINITION 5.4. The factor of quasiadjunction corresponding to κ is the polynomial $\Delta_\kappa = (t - \exp(2\pi i\kappa))(t - \exp(-2\pi i\kappa))$.

Now we are in position to formulate the main result.

THEOREM 5.1. *Let C be an irreducible plane algebraic curve such that all singularities have semisimple monodromy (e.g., unibranched singularities, nodes, etc.). Then*

$$A_1(\mathbf{R}) = \bigoplus_{\kappa} \bigoplus_{s_\kappa(d-3-\kappa d)} \mathbf{R}[t, t^{-1}]/(\Delta_\kappa) \tag{5.1}$$

where $\bigoplus_{s_\kappa(n)}$ denotes the direct sum of $s_\kappa(n)$ copies of the cyclic module $\mathbf{R}[t, t^{-1}]/(\Delta_\kappa)$. (**R** *is the field of reals.*)

COROLLARY 5.1. *Assume C has as singularities only the cusps and nodes. Then*

$$A_1(Q) = \bigoplus_{s} Q[t, t^{-1}]/t^2 - t + 1 \tag{5.2}$$

where s is the superabundance of the system of curves of degree $d - 3 - d/6$ passing through the cusps of C.

Indeed, in the case of cuspidal curves, $\kappa = \frac{1}{6}$, $\Delta_{1/6} = t^2 - t + 1$ and $\Gamma(P^2, \mathcal{Q}_{1/6}(n))$ consists of the curves of degree n passing through the cusps.

COROLLARY 5.2. *Assume C has as singularities only the singularities given locally by $x^2 + y^5 = 0$. Then*

$$A_1(Q) = \bigoplus_{s} Q[t, t^{-1}]/(t^4 - t^3 + t^2 - t + 1) \tag{5.3}$$

where s is the superabundance of the system of curves of degree $d - 3 - 3d/10$ passing through the singularities of C.

For the curve with singularities as in Corollary 5.2, one has as constants of quasiadjunction $\frac{3}{10}$ and $\frac{1}{10}$. Because $A_1(\mathbf{R})$ given in Theorem 5.1 in fact can be defined over Q, we obtain $s_{3/10}(d-3-3d/10) = s_{1/10}(d-3-d/10)$. Curves in the linear system $\Gamma(P^2, \mathcal{E}_{3/10}(n))$ are the curves of degree n just passing through the singularities of C. (Note that $\Gamma(P^2, \mathcal{E}_{1/10}(n))$ consists of the curves of degree n which are passing through the singularities of C and whose tangents at the singular points of C coincide with the tangents of C.)

EXAMPLE 5.2. Let C_{6k} (resp. C_{10k}) be given in the equation

$$f_{2k}^3 + f_{3k}^3 = 0 \tag{5.4a}$$

$$\left(\text{resp. } f_{2k}^5 + f_{5k}^2 = 0\right) \tag{5.4b}$$

where f_m denotes the defining polynomial of a nonsingular curve of degree m. Then C_{6k} (resp. C_{10k}) has as singularities only cusps (resp. only the singularities given locally by $x^2 + y^5 = 0$).

They are located at the intersection points of the curves $f_{2k} = 0$ and $f_{3k} = 0$ (resp. $f_{2k} = 0$ and $f_{5k} = 0$). By the Cayley-Bacharach theorem [3] the superabundance of the system of the curves of degree $5k - 3$ (resp. $7k - 3$) passing through the singularities of C_{6k} (resp. C_{10k}) is equal to 1. Hence the Alexander module is given by

$$A_1(Q) = \begin{cases} \text{for } C_{6k}: & Q[t, t^{-1}]/t^2 - t + 1, \\ \text{for } C_{10k}: & Q[t, t^{-1}]/t^4 - t^3 + t^2 - t + 1. \end{cases}$$

The proof of Theorem 5.1 will appear in [9]. Here we shall note that it uses the semisimplicity of the action of t on G'/G'' which follows from the proof of the divisibility theorem, and a combination of Lemma 4.1 with the appropriate generalization of Zariski's [19] arguments.

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