Proceedings of Symposia in Pure Mathematics
Volume 40 (1983), Part 2

DIFFERENTIABLE STRUCTURES ON COMPLETE INTERSECTIONS. II

ANATOLY S. LIBGOBER AND JOHN W. WOOD[1]

1. Introduction. In this paper we describe some work on the homotopy and diffeomorphism classification of nonsingular complete intersections. This is in part a survey of earlier results (especially **[LW2, LW3]**) and in part an extension of those results to the even-dimensional case. R. Thom observed that the diffeomorphism type of the complete intersection $X_n(\mathbf{d})$ is determined by the dimension n and the multidegree $\mathbf{d} = (d_1, \ldots, d_r)$. In fact these invariants determine the isotopy class of the embedding of X in CP_{n+r}. A basic problem is to describe X up to homotopy or diffeomorphism in terms of a minimal set of invariants computed from n and $\mathbf{d}$. For example when $n = 1$, X is diffeomorphic to a connected sum of tori and the number of summands is determined by the Euler characteristic $e = d\{2 - \Sigma_{j=1}^{r}(d_j - 1)\}$ where $d = \Pi_{j=1}^{r} d_j$ is the total degree.

By the Lefschetz theorem on hyperplane sections the inclusion $X_n \hookrightarrow CP_{n+r}$ is an n-equivalence, that is $\pi_i X \to \pi_i CP_{n+r}$ is an isomorphism for $i < n$ and an epimorphism for $i = n$. From this and Poincaré duality it follows that the homology groups of X_n and CP_n are the same except that the middle-dimensional group, $H = H_n(X; Z)$, is free of rank generally much greater than 1. (The rank can be computed from the Euler characteristic.)

Intersection pairing (i.e., Poincaré duality) makes H a unimodular bilinear form space which is symmetric for n even and skew symmetric for n odd. Study of this form leads to connected sum decompositions of X, the first step in our study of differentiable structures. The skew symmetric case gives rise to a Kervaire invariant studied by several authors, see especially **[B3]**. Differentiable structures in this case are studied in part I of this paper **[LW3]**. Connected sum decomposition in the symmetric case is treated in **[LW2]**; we summarize these results in §2.

1980 *Mathematics Subject Classification.* Primary 57R55, 14M10.
[1]The authors were partially supported by the National Science Foundation.

0082-0717/82/0000-0167/$03.50

Circumstances in which the homotopy type of X is determined by the integral cohomology ring are presented in §3. For this in §4 we construct degree one maps between nonsingular complete intersections from certain algebraic maps of X to projective space which are one-to-one with singular image, generalizing the notion of cuspidal projection for curves. The application of surgery theory is given in §5 and in §6 results on diffeomorphism, isotopy, and the moduli space of complete intersections are described.

Unless otherwise specified we will always assume $n > 2$; most of our methods do not work when $n = 2$.

The case $n = 3$ illustrates the program we would like to carry out in higher dimensions. A sequence of papers [**W, J, Z**] gives a complete classification of simply connected 6-manifolds up to homotopy or diffeomorphism type. For a complete intersection there is a smooth connected sum decomposition

$$X_3(\mathbf{d}) = M\#S^3 \times S^3\# \cdots \#S^3 \times S^3$$

where M is simply connected and has the same homology module as CP_3. If $x \in H^2(X; Z)$ is a generator, $x^3 \cap [M] = d$ is the total degree. The homotopy type of M is determined by d, the Stiefel-Whitney class w_2, and when d and w_2 are both even, the Pontryagin class $p_1 \bmod 48$. The diffeomorphism type of M is determined by d, w_2, and p_1; see [**LW3**, §9] for more detail. We have used a computer search to find examples of homotopy equivalence and of diffeomorphism:

$$X_3(15, 14, 3, 3, 2) \simeq X_3(18, 7, 6, 5)$$

and

$$X_3(16, 10, 7, 7, 2, 2, 2) = X_3(14, 14, 5, 4, 4, 4)$$

are the examples in each case with smallest Chern class c_1.

2. Connected sum decompositions. We begin by recalling results which permit us to decompose X up to diffeomorphism as a connected sum $X = M\#N$ where N is a smooth, $(n-1)$-connected, almost parallelizable manifold and where rank H_nM is as small as possible.

First let us consider the case when n is odd. Then the minimal rank of H_nM is either 2 or 0. The former case holds if and only if the following two circumstances happen.

(a) There is no homologically trivial n-sphere embedded in X with nontrivial normal bundle.

In this case one can construct a natural nondegenerate quadratic form on $H_n(X; Z/2)$ associated with the intersection pairing.

(b) The Arf invariant of this quadratic form is nontrivial.

In terms of the multidegree of the complete intersection, (a) holds if and only if the binomial coefficient $\binom{m+l}{m+1}$ is even where $n = 2m + 1 \neq 1, 3$, or 7 and l is the

number of even entires in $\mathbf{d} = (d_1, \ldots, d_r)$. Provided (a) holds, the Arf invariant, called the Kervaire invariant $K(X)$, is given as follows:

(i) If d is odd

$$K(x) = \begin{cases} 0 & \text{if } d \equiv \pm 1 \bmod 8, \\ 1 & \text{if } d \equiv \pm 3 \bmod 8. \end{cases}$$

(ii) If d is even $K(X) = 1$ if and only if $n \equiv 1 \bmod 8$, $l = 2$, and $8 \nmid d$.

This was proven by Browder [**B3**] with special cases (and alternate methods) in [**M, W1, W3, L, O**].

In the piecewise linear or topological category however one always has a decomposition $X \approx K\#N$ where rank $H_nK = 0$. The cohomology structure of K is given by $H^*(K; Z) = Z[x, y]/\{x^m = dy, y^2 = 0\}$, $n = 2m + 1$. We call a simply connected CW-space with this cohomology ring a d-twisted homology CP_n. For $d = 1$ such a space is homotopy equivalent to CP_n. We will call K a *core* and M a *smooth core* of X.

The situation for n even is quite different, rank H_nK can never be 0. The reason is that not all homology classes in H_nX are spherical. More precisely the image of the Hurewicz map $\pi_nX \to H_nX$ is the orthogonal complement in H_nX to the class h Poincaré dual to $x^{n/2}$ where x is the generator of $H^2(X)$. Geometrically h is the class of a section of X by a linear subspace of CP_{n+r} of dimension $n/2 + r$. The following results are proved in [**LW2**]:

(1) There is an orthogonal decomposition $H_nX = A \oplus B$ where $h \in A$ and rank $A \leqslant 5$.

(2) To any such homology decomposition corresponds a topological connected sum decomposition $X \approx K\#N$ such that $A = H_nK$, $B = H_nN$, and $N - D^{2n}$ is smooth, parallelizable, and $(n-1)$-connected.

But $\partial(N - D^{2n})$ is not necessarily a smooth S^{2n-1}. We call the manifold K corresponding to an A of minimal rank a *core* of X. It depends on choices made in its construction. The precise value of the minimal rank A, in fact the entire structure of A as a unimodular bilinear form space, is determined by d and the type. The type of A is the same as the type of H which is even iff the binomial coefficient $\binom{m+l}{l}$ is even where $n = 2m$ and l is the number of even entries in $\mathbf{d}$, see [**LW2**, §3].

Note that in fact $N = \alpha(S^n \times S^n)\#\beta(V \cup D^{2n})$ where V is the manifold obtained by plumbing of tangent bundles over S^n according to the graph E_8. Here α and β are determined by the Euler characteristic and signature of X and the rank and signature of A. For example if the intersection form on H has even type, then rank $A = 2$, $\alpha = \frac{1}{2}(e(x) - n - 2 - \sigma(x))$, and $\beta = \frac{1}{8}\sigma(X)$.

To construct a smooth connected sum decomposition of X let $\beta = q\beta_n + r$, $0 \leqslant r < \beta_n$, where β_n is the order of the group of homotopy spheres in dimension $2n - 1$. Since ∂V generates this group, $\beta_n(V \cup D^{2n}) = (V \natural \cdots \natural V) \cup D^{2n}$ is a smooth manifold and we have

(3) $X = M\#N$ where $N = \alpha(S^n \times S^n)\#q\beta_n(V \cup D^{2n})$ and $M = K\#r(V \cup D^{2n})$ are smooth.

M is called a *smooth core* of X; its homotopy type is determined by the homotopy type of the core K and by $\sigma(X)$.

3. Homotopy type. We are interested in the question under what circumstances the homotopy type of $X_n(\mathbf{d})$ or of a core K is determined by simple invariants. For example when n is odd K is a d-twisted homology CP_n. Assuming that d does not have small divisors it is shown in **[LW3, §2]** that K is homotopy equivalent to the $2n$-skeleton of the fibre of the map $CP_n \to K(Z/d, n+1)$. The result is

THEOREM 3.1. *If n is odd and if $p \mid d$ implies $2p \geq n+3$, then:*
(i) *Any two d-twisted homology CP_n's are homotopy equivalent.*
(ii) *The homotopy type of $X_n(\mathbf{d})$ is determined by n, d, and the Euler characteristic.*

Note that these invariants are all consequences of the integral cohomology ring. When n is even the core K is more complicated and we do not have a similar, obstruction theoretic characterization. Nevertheless we make the following

Conjecture. If $p \mid d$ implies $2p \geq n+3$, then the homotopy type of $X_n(\mathbf{d})$ and its core are determined by the integral cohomology ring.

This is true for n odd and also, for certain multidegrees, when n is even by 3.2 below. On the other hand, 3.4 gives an example where d has a small divisor in which cohomology operations distinguish between homotopy types. Under the assumption of the conjecture, however, by 3.5 cohomology operations are determined by the ring structure. We pose the question when the homotopy type or p-homotopy type is determined by the integral cohomology, for example assuming $2p \geq n+3$. A related fact due to Deligne, Griffiths, Morgan, and Sullivan is that any Kähler manifold is formal (see **[D]**) and in particular the rational homotopy type is determined by the rational cohomology. Pete Bousfield pointed out to us that if a finite complex is formal its p-homotopy type is determined for sufficiently large p.

W. Browder has made the appealing conjecture that the homotopy type of a complete intersection is determined by the integral cohomology ring and the type of the tangent sphere bundle as a spherical fibration.

We say a multidegree is pairwise relatively prime if d_i is prime to d_j for $i \neq j$.

THEOREM 3.2. *Let X and X' be two n-dimensional complete intersections with pairwise relatively prime multidegrees. Then they have homotopy equivalent cores if and only if they have the same total degree.*

PROOF. Note that since we assume $n > 2$, the total degree is determined by the cohomology ring of the core so the condition is necessary. Let $p_1^{e_1} \cdots p_k^{e_k}$ be the prime decomposition of the total degree of X, We will show that X has a core homotopy equivalent to a fixed core $\overline{K}$ of $\overline{X} = X_n(p_1^{e_1}, \ldots, p_k^{e_k})$. In the next section we construct a map $\varphi\colon X \to \overline{X}$ such that $\varphi^* \bar{x} = x$ where x and $\bar{x}$ are generators for $H^2(\,;Z)$. Since X and $\overline{X}$ have the same total degree, φ is a map of degree one.

The theorem then follows from

PROPOSITION 3.3. *Let $\varphi\colon X \to \overline{X}$ be a continuous map of degree one between complete intersections of dimension $n > 2$ which commutes up to homotopy with the inclusion in projective space and let $\overline{K}$ be a given core of $\overline{X}$. Then there is a core K of X which is homotopy equivalent to $\overline{K}$.*

PROOF. Let $\bar{\pi}\colon \overline{X} \to \overline{K}$ be the map collasping $\overline{N}$ to a point. If we split X as $K \# N$ where $\bar{\pi} \circ \varphi$ can be factored as $\bar{\varphi} \circ \pi$,

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & \overline{X} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \bar{\pi}} \\ K & \xrightarrow{\bar{\varphi}} & \overline{K} \end{array}$$

so that $\bar{\varphi}$ induces an isomorphism on homology, then by Whitehead's theorem $\bar{\varphi}$ will be a homotopy equivalence.

Suppose first that n is even. Since $\bar{\pi} \circ \varphi$ has degree one, Poincaré duality defines a splitting $\beta\colon H_*\overline{K} \to H_*X$ of $\bar{\pi} \circ \varphi$, cf. **[B1, 1.2.5]**. It follows from the definition of β that $\beta(h_{\bar{x}}) = h_x$ so we may take $A = \beta H_n \overline{K}$ as the unimodular summand of H_nX containing h_x and apply (2) of §2 to obtain a connected sum decomposition $X \approx K \# N$. Now $N - D^{2n}$ has the homotopy type of a bouquet of n-spheres. Such a sphere, $f\colon S^n \to X$, satisfies $f_*[S^n] \in A^{\perp}$, so $\varphi_* f_*[S^n] \in (H_n\overline{K})^{\perp}$. Since $\pi_n\overline{X}$ is a subgroup of $H_n\overline{X}$ **[W2, Lemma 2]** it follows that $\bar{\pi} \circ \varphi \circ f$ is null-homotopic so that $\bar{\pi} \circ \varphi$ can be factored as required above.

Now suppose n is odd. Then $\pi_n X = Z/d \oplus H_n X$ and similarly for $\overline{X}$; further $\varphi_{\#}$ restricts to an isomorphism on Z/d. Then $\pi_n\overline{K} = Z/d$ and $(\pi \circ \varphi)_{\#}$ restricts to an isomorphism. Hence the symplectic basis for H_nX given by embedded copies of $S^n \vee S^n$ in X used in the handle removing argument of **[W1, §1]** can be modified so that these spheres lie in $\ker(\bar{\pi} \circ \varphi)_{\#}$. This basis gives rise to a decomposition $X \approx K \# N$ such that $\bar{\pi} \circ \varphi$ factors through K as required. This completes the proof of Proposition 3.3.

The rest of §3 is not essential for the sequel; it concerns when cohomology operations can be used to distinguish homotopy types of cores and when they are formal consequences of the ring structure.

PROPOSITION 3.4. *If $2p \leq n+2$, $X_n(p, p)$ and $X_n(p^2)$ do not have homotopy equivalent cores.*

PROOF. We will show these cores are distinguished by their mod p characteristic class q_1, see **[MS, p. 229]**. First there is a Wu formula characterizing q_1 in terms of the action of the mod p Steenrod algebra, hence q_1 is a homotopy invariant. Next $\pi^* q_1 K = q_1 X$ where $\pi\colon X \to K$ is the collapsing map since q_1 is determined by the Pontryagin classes $p_1, \ldots, p_l$ ($l = (p-1)/2$) and these correspond under π^* for $2(p-1) < 2n$ since N is almost parallelizable. Hence $q_1(K)$ is determined by $q_1(X)$. Finally q_1 belongs to a multiplicative sequence in the Pontryagin classes

and so can be computed from its value on line bundles: $q_1(\gamma^{\otimes d}) = (dx)^{p-1}$. It follows tht $q_1(X_n(p, p)) = (n + 3 - 2p^{p-1})x^{p-1} \bmod p$ and $q_1(X_n(p^2)) = (n + 2 - p^{2(p-1)})x^{p-1} \bmod p$. For $2p \leqslant n + 2$, the class x^{p-1} is indivisible, hence these classes are different mod p.

For $p = 2$ replace q_1 by w_2 in the argument. Except when $p = 2$ and $n \equiv 2$ or 4 mod 8, the cores have isomorphic cohomology rings. (In the exceptional case the type of the intersection pairing is different.)

This method of detecting counterexamples cannot work when d does not have small divisors:

THEOREM 3.5. *If the total degree d of $X_n(\mathbf{d})$ satisfies $p \mid d \Rightarrow 2p \geqslant n + 3$, then the action of the Steenrod algebra (for any prime) is determined by the cohomology ring of X.*

PROOF. If n is odd this follows from 3.1 so we may assume n is even. Let E be the subgroup of $H^n(X; Z)$ dual to the vanishing cycles, $E = h^{\perp}$. The subgroup generated by $x^{n/2}$ and E has index d in $H^n(X; Z)$. Fix a prime p. Since $\dim x = 2$, $P^j x^k = \binom{k}{j} x^{k+j(p-1)}$ (see [SE, p. 78]) and we need only consider the action of P^j on $H^n(X; Z/p)$. The only primes p for which $P^j(H^n)$ lies in a nonzero group satisfy $2p \leqslant n + 2$ and these primes do not divide d. Therefore $x^{n/2}$ and E generate $H^n(X, Z/p)$. But the Steenrod operations vanish on E by the following, since X is a hyperplane section.

LEMMA 3.6. *Let V_n be a nonsingular hyperplane section and let E be the subgroup of $H^n(V)$ dual to the vanishing cycles. Then the Steenrod operations vanish on E.*

PROOF. E is generated by classes dual to homology classes represented by embedded spheres with normal bundle isomorphic to their tangent bundle [AF]. Let $f\colon S^n \hookrightarrow V$ be dual to $\alpha \in H^n(V)$, let N be the normal bundle of f, and N^* the Thom space. Let $p\colon V \to N^*$ denote the map which collapses the complement of N to the base point. Then $p^*(u) = \alpha$ where u generates $H^n(N^*)$. By naturality it is enough to show the Steenrod operations vanish on u. This follows from the fact that N^* is homeomorphic to the product $S^n \times S^n$ modulo the diagonal.

4. Special projections. The existence of the degree one map between complete intersections needed for 3.2 is a consequence of the following

LEMMA 4.1. *If a and b are relatively prime, then there is a degree one map*

$$\varphi\colon X_n(ab, c_1, \ldots, c_k) \to X_n(a, b, c_1, \ldots, c_k).$$

PROOF. By repeated use of Bertini's theorem the hypersurfaces given by the equations

$$z_0^a + \cdots + z_m^a = 0,$$

$$\lambda_0 z_0^b + \cdots + \lambda_m z_m^b = 0, \qquad \lambda_0 \neq 0,$$

and k equations of the form

$$f_i = \mu_1 z_1^{c_i} + \cdots + \mu_m z_m^{c_i} = 0, \qquad i = 1, \ldots, k,$$

have nonsingular intersection for generic choice of the coefficients. Bertini's theorem applies since the only base point of such a system, $[1, 0, \ldots, 0]$, does not satisfy the first equation. Thus we may represent $X_n(a, b, c_1, \ldots, c_k)$ up to diffeomorphism by this complete intersection, cf. [W3, §2].

The image of this variety under projection to CP_{m-1} from $[1, 0, \ldots, 0]$, that is under the map sending $[z_0, \ldots, z_m]$ to $[z_1, \ldots, z_m]$, is given by the equations

$$(-z_1^a - \cdots - z_m^a)^b = \left(-\frac{\lambda_1}{\lambda_0} z_1^b - \cdots - \frac{\lambda_m}{\lambda_0} z_m^b\right)^a$$

and $f_i = 0$ for $i = 1, \ldots, k$. The image is a singualr variety V. If a given point $[z_1, \ldots, z_m] \in V$ is the image of $[z_0, z_1, \ldots, z_m]$, then z_0^a and z_0^b are determined by $(z_1, \ldots, z_m)$. Since a and b are relatively prime, z_0 is determined. Therefore π: $X_n(a, b, c_1, \ldots, c_k) \to V$ is 1-1 and hence a homeomorphism.

Let ψ be the retraction induced by degeneration of a nonsingular complete intersection $X_n(ab, c_1, \ldots, c_k)$ onto V. Then $\varphi = \pi^{-1} \circ \psi$. These maps commute with inclusion into CP_m so the generators of $H^2(\ ; Z)$ correspond. Since the total degrees are the same, φ has degree one.

Another consequence of this construction is

COROLLARY 4.2. *If $\mathbf{d} = (d_1, \ldots, d_r)$ is pairwise relatively prime then there is a 1-1 algebraic map of $X_n(\mathbf{d})$ onto a singular hypersurface V of degree d. In particular $X_n(\mathbf{d})$ and V are homeomorphic.*

Given $X_n(\mathbf{d}) \subset CP_{n+r}$ the question of algebraically embedding X in lower codimension has been much studied. Very recently L. Astey and S. Gitler have obtained nonimmersion results in the smooth category. In the case of curves, 1-1 algebraic maps are called cuspidal projections and have been studied by R. Piene [P]. §3 implies necessary conditions for such maps to exist.

PROPOSITION 4.3. *Let $\overline{X} = X_n(\mathbf{d})$ be a complete intersection which admits a 1-1 algebraic map into CP_{n+1}. Then $\overline{X}$ and the hypersurface $X = X_n(d)$ have homotopy equivalent cores.*

PROOF. The image of $\overline{X}$ in CP_{n+1} is a singular hypersurface V and the map $\overline{X} \to V$ is a homeomorphism. The retraction of X on V provides a continuous, degree one map $X \to \overline{X}$ to which Proposition 3.3 applies. Hence the cores are homotopy equivalent.

As a result applying 3.4 we get

COROLLARY 4.4. *If $2p \leqslant n + 2$, $X_n(p, p)$ does not admit a 1-1 algebraic map into CP_{n+1}.*

5. Application of surgery theory. In this section we apply the exact sequence of surgery theory to a smooth core of an even-dimensional complete intersection. The treatment here is analogous to that in [LW3, §§3 and 4] for the case when M

is a d-twisted homology CP_n for n odd. The main consequence is

THEOREM 5.1. *The number of distinct diffeomorphism classes of complete intersections of even dimension n and total degree d with a core K of a fixed homotopy and with given Euler characteristic and Pontryagin classes is less than a bound depending only on n.*

Let M be a fixed smooth core and denote by $hS(M)$ the set of smooth manifolds homotopy equivalent to M.

LEMMA 5.2. *If the middle Betti number of M is $\leq b$, the number of elements of $hS(M)$ with a given set of Pontryagin classes is bounded by a function of n and b.*

PROOF. The exact sequence of surgery theory gives an inclusion of $hS(M)$ in $[M, G/O]$, see [**B1**, II4.10]. There is also an exact sequence

$$\tilde{\pi}_s^0(M) \to [M, G/O] \to \widetilde{KO}^0(M)$$

where the first term is reduced stable cohomotopy and the last is reduced real K-theory, see [**LW3**, §3]. (This sequence is induced from the sequence of fibrations $SG \to G/O \to BSO$.) If the homotopy equivalence $f\colon M_1 \to M$ represents an element of $hS(M)$, its image in $KO^0(M)$ is given by $f^{-1*}\tau M_1 - \tau M$. If M_1 and M have the same Pontryagin classes, then this image lies in the torsion subgroup of $KO^0(M)$, cf. [**B2**, Lemma 2.24]. Also the set of elements in $[M, G/O]$ with the same image in $KO^0(M)$ is bounded by the order of $\tilde{\pi}_s^0(M)$. Thus Lemma 5.2 follows from

LEMMA 5.3. *Let M be a core of a complete intersection of even dimension n with middle Betti number less than or equal to b. Then:*

(i) *$\pi_s^0(M)$ is a finite group of order bounded by a function of b and n (and independent of the degree).*

(ii) *$KO(M)$ has only even torsion of order $\leq 2^b$.*

PROOF. The first assertion follows immediately from the Atiyah-Hirzebruch spectral sequence for stable cohomotopy. The bound depends on b and the orders of the stable homotopy groups of spheres in dimensions $\leq 2n$, see [**LW3**, Theorem 3.3(ii)].

Since $H^*(M; Z)$ is free and concentrated in even dimensions, the corresponding spectral sequence for KO-theory shows that $KO(M)$ has no odd torsion. Replacing the map $f\colon M \to CP_n$ by an inclusion the cohomology of the pair is given by

$$H^q(CP_n, M; Z) = \begin{cases} \text{free of rank} \leq b-1, & q = n+1, \\ Z/d, & q = n+3, n+5, \ldots, 2n+1, \\ 0, & \text{otherwise.} \end{cases}$$

It follows from the spectral sequence for the pair that the even torsion in $KO^1(CP_n, M)$ is at most of order 2^{b-1}. Since the torsion in $KO^0(CP_n)$ has order at most 2, assertion (ii) follows.

For the odd-dimensional case in [**LW3**, §§4 and 5] with additional assumptions more precise bounds are obtained.

PROOF OF 5.1. For any X in a set of complete intersections with fixed invariants as described in the theorem let $X = M \# N$ be a differentiable connected sum decomposition where M is a minimal smooth core. The Pontryagin classes of X determine the signature so by §2 the diffeomorphism type of N is the same for each X as is the homotopy type of M. The middle Betti number of M is bounded by $b = 5 + 8\beta_n$ which depends only on n. The Pontryagin classes of M are determined by those of X for if $\pi\colon X \to M$ is the collapsing map, $\pi^*\colon H^*(M; Z) \to H^*(X; Z)$ is injective and $\pi^* p_j(M) = p_j(X)$ for $2j < n$ since the summand N is almost parallelizable. The top Pontryagin class is then determined by the signature, $\sigma(M) = \sigma(X) - \sigma(N)$. Therefore the theorem follows from Lemma 5.2.

6. Diffeomorphic complete intersections, isotopy, and moduli spaces. As a consequence of the results of §§3 and 5 we have the following

THEOREM 6.1. *In any dimension $n \neq 2$ and for any integer k there are k distinct multidegrees for which the corresponding complete intersections are all diffeomorphic.*

PROOF. For n odd see [**LW3**, 6.3]. We give here the proof for n even. It suffices to produce a sufficiently large number of complete intersections with the same invariants as required in Theorem 5.1. The key result is a counting argument due to A. O. L. Atkin, see [**LW3**, §6] for the proof.

PROPOSITION 6.2. *Given integers n and N there are integers r and d such that the number of distinct unordered r-tuples $(d_1, \ldots, d_r)$ with product d and with the same first n symmetric functions is greater than N. Moreover d may be taken to be a product of $2r$ distinct primes.*

It follows from [**LW3**, 6.1] that the Euler characteristic and Pontryagin classes of the corresponding n-dimensional complete intersections are the same. Since each multidegree is pairwise relatively prime, the cores are homotopy equivalent by 3.2. The result follows by taking N greater than k times the bound in 5.1.

For $n = 2$ the recent work of Mike Freedman shows that the homeomorphism type of these manifolds is determined by the homotopy type and hence there are many homeomorphic complete intersections with different multidegrees.

We conclude by quoting some results from [**LW3**] which hold for both even and odd n and which relate equivalence up to diffeomorphism to isotopy and analytic equivalence. First, examples of 5.1 never exist with small codimension because in that case the differential structure on X determines the multidegree. More precisely we have

THEOREM 6.3. *If $r = \operatorname{codim}(X_n \subset CP_{n+r}) \leq n/2$, $n > 2$, then d and the Pontryagin classes of X_n determine certain symmetric functions of d from which r and $d_1, \ldots, d_r$ can be recovered. If r is given and $r \leq 1 + n/2$, $d_1, \ldots, d_r$ can still be recovered.*

An example shows the conditions on r are sharp: $X_3(12, 10)$ and $X_3(15, 4, 2)$ have the same degree and Pontryagin class, in fact $X_3(12, 10) = X_3(15, 4, 2) \# 13440(S^3 \times S^3)$. By 6.3 this would not be possible if both had codimension 2 or if either had codimension 1.

Using this result and Thom's observation for the low codimension case and Haefliger's work on isotopy classes of embeddings in higher codimension we deduce

THEOREM 6.4. *If X_n and Y_n are complete intersections in CP_{n+r} and if X_n is diffeomorphic to Y_n, then there is a diffeomorphism of CP_{n+r} isotopic to the identity carrying X_n to Y_n.*

On the other hand elementary considerations show

THEOREM 6.5. *If X_n and Y_n are complete intersections of dimension $n \geq 2$ (and with $c_1 \neq 0$ if $n = 2$) and if X_n is analytically equivalent to Y_n, then there is a projective linear transformation of CP_{n+r} carrying X_n to Y_n. Further they have the same multidegree.*

As was pointed out by B. Moishezon this implies

COROLLARY 6.6. *The moduli space of complex structures on the smooth manifold underlying a complete intersection can have arbitrarily many irreducible components.*

Indeed if X and Y are complete intersections in the same component they are connected by a sequence of small deformations. Then by a result of Sernesi [S] X would be analytically equivalent to a complete intersection of the same multidegree as Y so by 6.5 X and Y would have the same multidegree. Thus 6.1 implies the corollary.

REFERENCES

[AF] A. Andreotti and T. Frankel, *The second Lefschetz theorem on hyperplane sections*, Global Analysis, Princeton Univ. Press, Princeton, N.J., 1969, pp. 1–20.

[B1] W. Browder, *Surgery on simply connected manifolds*, Springer-Verlag, Berlin and New York, 1972.

[B2] ———, *Surgery and the theory of differentiable transformation groups*, Proc. Tulane Sympos. on Transformation groups, Springer-Verlag, Berlin and New York, 1968, pp. 1–46.

[B3] ———, *Complete intersections and the Kervaire invariant*, Lecture Notes in Math., vol. 763, Springer-Verlag, Berlin and New York, 1979, pp. 88–108.

[D] P. Deligne, P. Griffiths, J. Morgan and D. Sullivan, *Real homotopy theory of Kähler manifolds*, Invent. Math. **29** (1975), 245–274.

[J] P. Jupp, *Classification of certain 6-manifolds*, Proc. Cambridge Philos. Soc. **73** (1973), 293–300.

[L] A. Libgober, *Geometric procedure for killing the middle-dimensional homology groups of algebraic hypersurfaces*, Proc. Amer. Math. Soc. **63** (1977), 198–202.

[LW1] A. Libgober and J. Wood, *Diffeomorphic complete intersections with different multidegrees*, Bull. Amer. Math. Soc. (N. S.) **2** (1980), 459–461.

[LW2] ———, *On the topological structure of even dimensional complete intersections*, Trans. Amer. Math. Soc. **267** (1981), 637–660.

[LW3] ———, *Differentiable structures on complete intersections*. I, Topology **21** (1982), 469–482.

[MS] J. Milnor and T. Stasheff, *Characteristic classes*, Princeton Univ. Press, Princeton, N.J., 1974.

[M] S. Morita, *The Kervaire invariant of hypersurfaces in complex projective space*, Comment. Math. Helv. **50** (1975), 403–419.

[O] S. Ochanine, *Signature modulo 16, invariants de Kervaire généralisés et nombres caractéristiques dans la K-théorie réelle*, Mem. Soc. Math. France (N.S.) **5**.

[P] R. Piene, *Cuspidal projections of space curves*, Math. Ann. **256** (1981), 95–119.

[S] E. Sernesi, *Small deformations of global complete intersections*, Boll. Un. Mat. Ital. **12** (1975), 138–146.

[SE] N. Steenrod and D. B. A. Epstein, *Cohomology operations*, Princeton Univ. Press, Princeton, N.J., 1962.

[W] C. T. C. Wall, *Classification problems in differential topology*. V. *On certain 6-manifolds*, Invent. Math. **1** (1966), 355–374.

[W1] J. Wood, *Removing handles from nonsingular algebraic hypersurfaces in CP_{n+1}*, Invent. Math. **31** (1975), 1–6.

[W2] ———, *A connected sum decomposition for complete intersections*, Proc. Sympos. Pure Math., vol. 32, 2, Amer. Math. Soc., Providence, R.I, 1978, pp. 191–193.

[W3] ———, *Complete intersections as branched covers and the Kervaire invariant*, Math. Ann. **240** (1979), 223–230.

[Ž] A. V. Žubr, *Classification of simply connected six-dimensional spinor manifolds*, Math. U.S.S.R.-Izv. **9** (1975), 793–812.

UNIVERSITY OF ILLINOIS AT CHICAGO