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ALEXANDER MODULES OF PLANE ALGEBRAIC CURVES

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ABSTRACT. We introduce the Alexander modules for the plane algebraic curves and

1. Using this notion obtain some information on the torsion of the factor of the first commutator by the second commutator for the fundamental groups of plane algebraic curves.
2. Show that the second Alexander module is a free $Z[t,t^{-1}]$-module.
3. Prove a sufficient condition for a cyclic covering of CP^2 branched over a singular algebraic curve to be simply-connected.

As part of the study we obtain formulas for various homology invariants of coverings of CP^2 branched over immersed surfaces with non-locally flat points.

1. INTRODUCTION AND SUMMARY OF RESULTS

Let C be an algebraic curve in CP^2 which may possess arbitrary types of singularities. The basic problem in one- and two-dimensional algebraic geometry is to understand the complement to C in CP^2 (see [6]). In this paper we describe the invariants of CP^2-C derived from the homology of the cyclic coverings associated with C. They stem from the invariants which proved to be useful in the topological study of knotted spheres, another codimension 2 phenomenon ([14]). Part of our results are topological and are valid for any immersion of a closed orientable surface in CP^2. We shall show how the algebraicity of those immersions affects the invariants introduced.

Let L be the line at infinity which we shall assume to be transversal to C. We have $H_1(CP^2-(C\cup L),Z) = Z$ (see e.g., Lemma 1 below). This allows us to consider the uniquely defined infinite cyclic covering $\widetilde{CP^2-C\cup L}$ of the affine portion $CP^2-(C\cup L)$ of the curve C. Let R be a commutative ring. We define the Alexander modules $A_i(R)$

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of the curve C (or of the immersed orientable surface) as the homology groups $H_i(CP^2\text{-}CUL,R)$ considered as the module over the group ring $R[Z]$.

The action of Z on H_1 is obtained via the action on $CP^2\text{-}CUL$ by deck transformations. The modules $A_i(R)$ provide an invariant of the complement of C and in particular carry the information about finite cyclic coverings of CP^2 branched along C and possibly L. These results are discussed in Section 2.

In [15] it has been shown that $A_1(Q)$ is a $Q[Z]$-torsion module and the order $\Delta_C(Q)$ of $A_1(Q)$ was called the Alexander polynomial of C. This is an element in $Q[t,t^{-1}]$ defined uniquely up to a unit of this ring. Here we consider $\Delta_C(F)$ for arbitrary field F. The Alexander polynomial $\Delta_C(F)$ has the following divisibility properties -- the proof of which coincides with the proof of the theorems 1 and 2 from [15], with the only change of notation the field $\mathbb{C}$ to the arbitrary field F.

THEOREM 1: Let C be a plane algebraic curve with arbitrary singularities. Let $\Delta_{p_i,C}(F)$ denote the Alexander polynomial of the link of a singular point p_i. Let $\Delta_\infty(F)$ denote the Alexander polynomial of the link $C \cap \partial T(L) \subset \partial T(L)$ where $\partial T(L) = S^3$ is a sufficiently small tubular neighborhood of the line L in infinity.

(The coefficients of all Alexander polynomials are taken in the field F (cf. [24])). Then

1. $\Delta_C(F)$ divides $\Pi\Delta_{p_i}(F)$ where p_i runs over all branches of all singular points of C.
2. $\Delta_C(F)$ divides $\Delta_\infty(F)$.

In particular this theorem provides the information on the torsion of $A_1(Z) = \pi_1(CP^2\text{-}C)'/\pi_1(CP^2\text{-}C)'' = \pi_1(CP^2 - CUL))'/\pi_1(CP^2\text{-}(CUL))''$ where G' and G'' denote the first and second commutators of the group G. (Note that the latter isomorphism follows from Zariski's lemma on the relation between $\pi_1(CP^2\text{-}C)$ and $\pi_1(CP^2\text{-}(CUL))$ (see [25] or [15], Section 7). Here is an example of such information for cuspidal curves. We leave to the interested reader to derive similar corollaries in the case of other singularities.

COROLLARY 1.1: Let C be a curve which admits as singularities only cusps and nodes. Then

1) If C has an odd degree not divisible by 3, then $\pi_1(CP^2\text{-}C)'$ is a perfect group.

2) If C has an odd degree not divisible by 3, then $\pi_1(CP^2\text{-}C)'/\pi_1(CP^2\text{-}C)''$ has only 2-torsion.

3) If C has an even degree not divisible by 6, then $\pi_1(CP^2\text{-}C)'/\pi_1(CP^2\text{-}C)''$ is a 3-torsion group.

Note that Zariski in [26], p. 618, and in [27], p. 358, showed that $\pi_1(CP^2\text{-}C)'$ is trivial for some curves of odd degree of genus zero and one. On the other hand for the three-cuspidal quadric we have $A_1(Z) = Z_3$ (cf. [16]). The Case 1 of this corollary also follows from the R. Randell work ([23]).

PROOF OF THE COROLLARY: The local Alexander polynomials for curves with nodes and cusps are $\Delta_n(F) = t-1$ for each branch of a node and $\Delta_c(F) = t^2-t+1$ for a cusp. Because we assume that L is in a general position we have $\Delta_\infty(F) = (t-1)(t^d-1)^{d-2}$ where d is the degree of C ([15], Section 6). If the characteristic of F_p is not 2 or 3 then $(t^2-t+1)^N$ (N denotes the number of cusps) and $(t-1)(t^d-1)^{d-2}$ are relatively prime provided d is not divisible by 6. The most simple-minded way to see this is to observe that the existence of common roots of t^d-1 and t^2-t+1 depends only on $d \bmod 6$ and then to examine five cases $d = 1,\ldots 5$. This implies that if $d \equiv \pm 1(6)$ then t^d-1 and t^2-t+1 are relatively prime in any characteristic, i.e., $\Delta_C(Z/pZ) = 1$ and hence $A_1(Z)$ has no p-torsion for any p. If $d \equiv 3(6)$ then t^2-t+1 and t^d-1 are relatively prime for any $p \neq 2$. If $d \equiv \pm 2(6)$ then t^2-t+1 and t^d-1 are relatively prime for any $p \neq 3$.

Now we consider the structure of $A_2(Z)$ and $A_3(Z)$. It is given by the following

THEOREM 2: Let C be a plane algebraic curve of degree d with arbitrary singularities. Then

1. $A_2(Z)$ is the free $Z[t,t^{-1}]$ module of the rank equal to $\text{rk}\, H_1(C) + d - 1$.
2. $A_3(Z) = 0$.

Note that part 2 is also valid for any immersion of an orientable surface in CP^2. However the algebraicity of C is essential in part 1. Theorem 2 is proven in Section 3.

In Section 4 we prove the following

THEOREM 3: Let F_k be the cyclic k-fold branched covering of CP^2 branched over C and line L. Let $\tilde{F}_k$ be a desingularization of F_k and Σ be the singular set of F_k. Let T be the $Z[t,t^{-1}]$-submodule of $A_1(Z) = H_1(CP^2\text{-}CUL,Z)$ of Z-torsion elements. Let $T_k = T/(t^k-1)T$. Let ω_k be a primitive root of unity of order k.

1. If $\Delta_C(\omega_k) \neq 0$, then $H_1(F_k-\Sigma,Z)$ is finite and

$$\text{order } H_1(F_k-\Sigma,Z) = \prod_{i=1}^{k-1} \Delta_C(\omega^i) \quad (\text{order } T_k) \tag{1.1}$$

2. If all singularities of C are either unibranched or nodes and if ω_k is a root of none of the Alexander polynomials of singularities of C then $\tilde{F}_k$ is simply-connected.

Note that (1.1) is closely related to the formula (10) from [28]. Part 2 implies a stronger result than the main conclusion of [28] where triviality of $H_1(F_{p^k},Q)$ is shown.

COROLLARY: If all singularities of C are either unibranched or nodes then the desingularization of the covering of CP^2 of a degree which is a power of prime is simply connected.

PROOF follows from the Theorem 3.2 and from the observation that a polynomial f with integer coefficients such that $f(1) = 1$ can not as zero have a primitive root of prime power (cf. [28], p. 498).

We conclude with Section 5 which contains relevant remarks and examples of the computations of Alexander module.

2. ALEXANDER MODULES OF IMMERSED SURFACES IN CP^2

Let C be an orientable surface of real dimension 2 immersed in CP^2 with possibly several non-locally flat points. Assume that C represents in $H_2(CP^2,Z) = Z$ the homology class corresponding to integer d. Let L be a fixed line CP^1 which we shall assume transveral to C. This assumption is made for simplicity. Everything that follows can be modified to include the non-transversal case as well ([15]). Of course our main concern is the case when C is an algebraic curve, i.e., the set of zeroes of a polynomial equation. However in this section this is irrelevant except for the last lemma.

We have the following

LEMMA 2.1: The reduced homology of CP^2-(CUL) are given by

$$H_i(CP^2\text{-}(CUL),Z) = \begin{cases} Z & i = 1 \\ Z^{b} & i = 2 \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

where $b = \text{rk } H_1(C) + d-1$.

PROOF is a routine use (cf. [15]) of the exact sequence of the pair $(CP^2, CP^2\text{-}(CUL))$, of the Lefshetz duality and of the fact that

$$H^i(CUL,Z) = \begin{cases} Z^b & i = 1 \\ Z^2 & i = 2 \end{cases} \tag{2.2}$$

Let $\widetilde{CP^2\text{-}CUL}$ denote the infinite cyclic covering of CP^2-(CUL) uniquely defined by Lemma 2.1. Let R be an arbitrary commutative ring. The group Z acts on $\widetilde{CP^2\text{-}CUL}$ and consequently on $H_i(\widetilde{CP^2\text{-}CUL},R)$, which makes this group a $R[Z]$-module. We denote the ring $R[Z]$ by $\Lambda(R)$. This is the ring of Laurent polynomials with R-coefficients.

DEFINITION 2.1: The Alexander module $A_i(R)$ of the surface C with coefficients in R is the homology group $H_i(\widetilde{CP^2\text{-}CUL},R)$ considered as a $\Lambda(R)$-module.

Note that if $i \geq 4$ then $A_i(R) = 0$ for a trivial reason.

If R is a field then $\Lambda(R)$ is a principal ideal domain. Hence any finitely generated module is a direct sum of cyclic modules.

LEMMA 2.2: The rank of the $\Lambda(R)$-free part of $A_2(R)$ is equal to $b = \text{rk } H_1(C) + d-1$. The rank of the $\Lambda(R)$-free part of $A_3(R)$ is zero.

PROOF follows by the standard application of Milnor's argument ([17]). The exact sequence of the chain complexes

$$0 \to C(\widetilde{CP^2-CUL}) \xrightarrow{t-1} C(\widetilde{CP^2-CUL}) \to C(P^2-CUL) \to 0 \quad (2.3)$$

(t is induced by the deck transformation) together with the Lemma 2.1 gives

$$0 \to H_3(\widetilde{CP^2-CUL},R) \xrightarrow{t-1} H_3(\widetilde{CP^2-CUL},R) \to 0 \to H_2(\widetilde{CP^2-CUL},R)$$
$$\xrightarrow{t-1} H_2(\widetilde{CP^2-CUL},R) \to R^b \to H_1(\widetilde{CP^2-CUL},R) \xrightarrow{t-1} \quad (2.4)$$
$$\to H_1(\widetilde{CP^2-CUL},R) \to R \to R \xrightarrow{0} R.$$

This implies that $A_3(R)$ cannot have a free summand or (t-1)-torsion summands. $A_2(R)$ cannot contain (t-1)-torsion summands because multiplication by (t-1) is injective on $A_2(R)$. Hence the rank of the free part of $A_2(R)$ is b. Q.E.D.

Let λ_j^i be defined from the cyclic decomposition of Alexander modules:

$$A_1(R) = \bigoplus_{j=1}^{h_1} \Lambda/\lambda_j^1, \quad A_2(R) = \oplus\Lambda^b \bigoplus_{j=1}^{h_2} \Lambda/\lambda_j^2, \quad A_3 = \bigoplus_{j=1}^{h_3} \Lambda/\lambda_j^3 \quad (2.5)$$

Let

$$\Delta^i(R) = \prod_{j=1}^{h_i} \lambda_j^i . \quad (2.6)$$

It follows from the proof of the Lemma 2.2 that we can normalize $\Delta^i(R)$ in such a way that $\Delta^i(R)(1) = 1$. $\Delta^1(Q)$ is the Alexander polynomial considered in [15].

The next lemma represents the relation imposed by the Blanchfield duality ([14]) existing in any infinite cyclic covering.

Recall that for any $\Lambda(R)$ module A the dual module structure is defined by $\lambda\cdot x = x\cdot\bar{\lambda}$ where $\bar{\lambda}(t) = \lambda(t^{-1})$.

LEMMA 2.3: In the notation above

1. $\Delta^2(R)$ divides $\overline{\Delta^1(R)}$
2. $\Delta^3(R) = 1$, i.e., $A_3(R) = 0$ for any field R.

PROOF: Let X be the complement to the regular neighborhood of CUL. Let ∂X denote the boundary of the manifold X. Let $\tilde{X}$ and $\partial\tilde{X}$ be the infinite cyclic coverings. Then we have the duality relation ([14])

$$H_i(\tilde{X},R) = \overline{H_e^{4-i}(\tilde{X},\partial\tilde{X})} \quad (2.7)$$

where H_e is the cohomology group of the chain complex $\mathrm{Hom}_{\Lambda(R)}(C(\tilde{X},\partial\tilde{X}),\Lambda(R))$. The universal coefficients theorem implies

$$\text{Free part } H_i(\tilde{X}) = \text{Free part } \overline{H_{4-i}(\tilde{X},\partial\tilde{X})}$$
$$(2.8)$$
$$\text{Torsion part } H_i(\tilde{X}) = \text{Torsion part } \overline{H_{3-i}(\tilde{X},\partial\tilde{X})}.$$

The latter isomorphism implies

$$H_3(\widetilde{CP^2-CUL},R) = H_0(\widetilde{CP^2-CUL}, \partial(\widetilde{CP^2-CUL}),R) = 0$$

which gives part 2 of the lemma.

Now the end of the exact sequence of the pair

$$H_1(\widetilde{CP^2-CUL},R) \to H_1(\widetilde{CP^2-CUL}, \partial(\widetilde{CP^2-CUL}) \to 0$$

can be written using (2.8) as

$$\text{Tor } H_1(\widetilde{CP^2-CUL},R) \to \text{Tor } \overline{H_2(\widetilde{CP^2-CUL},R)} \to 0. \quad (2.9)$$

This implies part 1 of the lemma. Q.E.D.

The universal coefficients theorem implies

COROLLARY 2.1: $A_3(Z) = 0$.

In the rest of this section we discuss the finite coverings of CP^2 associated with CUL.

LEMMA 2.4: Let $(\widetilde{CP^2-CUL})_k$ denote the cyclic covering of CP^2-CUL defined by the homomorphism $H_1(CP^2\text{-}CUL,Z) = Z \to Z/kZ$. Let c_k^i be the sum over j of the number of the common roots in the algebraic closure $\bar{R}$ of R of t^k-1 and λ_k^i. Then

$$\operatorname{rk} H_i((CP^2-CUL)_k,R) = \begin{cases} c_k^1 + 1 & i = 1 \\ kb + c_k^1 + c_k^2 & i = 2 \\ c_k^2 & i = 3 \end{cases} \qquad (2.10)$$

PROOF. We follow the scheme of Summers [21]. Clearly it is enough to check the claim about Betti numbers for $\bar{R}$-coefficients. The exact homology sequence, corresponding to the chain complex

$$0 \to C(\widetilde{CP^2}-CUL) \xrightarrow{t^k-1} C(\widetilde{CP^2}-CUL) \to C((CP^2-CUL)_k) \to 0$$

implies

$$\begin{aligned} 0 \to \operatorname{Coker}(H_i(\widetilde{CP^2}-CUL) &\xrightarrow{t^k-1} H_i(\widetilde{CP^2}-CUL)) \to H_i((CP^2-CUL)_k) \to \\ \to \operatorname{Ker}(H_{i-1}(\widetilde{CP^2}-CUL) &\xrightarrow{t^k-1} H_{i-1}(\widetilde{CP^2}-CUL)) \to 0 \end{aligned} \qquad (2.12)$$

Representing $H_i(CP^2-CUL),\bar{R})$ as a sum of cyclic modules we see (using Lemma 2.2) that

$$\operatorname{rk} \ker(H_i(\widetilde{CP^2}-CUL,\bar{R}) \xrightarrow{t^k-1} H_i(\widetilde{CP^2}-CUL),R)) = \begin{cases} c_k^1 & i = 1 \\ kb + c_k^2 & i = 2 \end{cases} \qquad (2.14)$$

Clearly (2.12)-(2.14) implies the lemma. Let F_k denote the cyclic branched covering of CP^2 branching over CUL ([15], [8]). We denote by CUL the preimage of CUL in F_k. Let Σ be the set of singular points of F_k (i.e., preimages of non-locally flat points of C and points $C \cap L$). The following lemma relates the homology of unbranched and branched coverings.

LEMMA 2.5: Let $b_1 = (g,c,d(d,k)-1)(d-2) + 2(d-1) + \operatorname{rk} H_1(C)$ and $|\Sigma|$ be the number of points in Σ. Then we have the following exact sequences

$$0 \to H_3(F_k-CUL,Z) \to H_3(F_k-\Sigma,Z) \to Z^{|\Sigma|-1} \to 0 \qquad (2.15)$$

$$0 \to Z^{b_1-|\Sigma|+1} \to H_2(F_k-CUL,Z) \to H_2(F_k-\Sigma,Z) \to Z \to 0 \qquad (2.16)$$

$$0 \to Z \to H_1(F_k-CUL,Z) \to H_1(F_k-\Sigma,Z) \to 0 \qquad (2.17)$$

The last sequence is splitable.

PROOF: Let us consider the exact sequence of the closed subspace $CUL-\Sigma$ of $F_k-\Sigma$. We have

$$\begin{array}{l} 0 \to H_3(F_k-CUL,Z) \xrightarrow{j_3} H_3(F_k-\Sigma,Z) \xrightarrow{i_3} H_1(CUL-\Sigma,Z) \to \\ \xrightarrow{t_1} H_2(F_k-CUL,Z) \xrightarrow{j_2} H_2(F_k-\Sigma,Z) \xrightarrow{i_2} H_0(CUL-\Sigma,Z) \to \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \wr\!\wr\; i_2' \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad Z \oplus Z \\ \xrightarrow{t_0} H_1(F_k-CUL,Z) \xrightarrow{j_1} H_1(F_k-\Sigma,Z) \to 0. \end{array} \qquad (2.18)$$

Isomorphism i_2' follows from connectivity $\tilde{L}$. To show (2.16) it is enough to check that

(a) $\operatorname{rk} \operatorname{Im} i_2 = 1$ and $\operatorname{Im} i_2$ contains an indivisible element.

(b) There is the projection $\ell : H_1(F_k-CUL,Z) \to \operatorname{Coker} i_2 = Z$ such that $\ell \circ t_0$ is identity.

The first part of (a) was checked in Section 3 of [15]. The existence of an indivisible element follows from comparison (2.18) with the corresponding exact sequence for $(CP^2-CUL,\ CP^2-\Sigma)$ and from indivisibility L in $H_2(CP^2-\Sigma,Z)$. For part (b) we define projection $\ell : H_1(F_k-CUL,Z) \to \operatorname{Coker} i_2 = Z$ as the linking number of the image of 1-cycle in CP^2-L with 2 - cycle C-L. The linking number clearly is well-defined.

To show (2.15) and (2.16) we observe that $H_1(CUL-\Sigma,Z)$ is the free abelian group of rank b_1. Now (2.15) can be obtained just by comparison of the ranks of participating groups.

COROLLARY 1: In the notations of Lemma 2.4 we have

$$\operatorname{rk} H_i(F_k-\Sigma,Q) = \begin{cases} c_k^1 & i = 1 \\ c_k^2 + c_k^1 - b_1 + |\Sigma| & i = 2 \\ c_k^2 + |\Sigma| - 1 & i = 3 \end{cases}$$

PROOF follows from Lemmas 2.4 and 2.5.

The relations between the homologies of $F_k - \Sigma$ and F_k on one side and the homologies of $F_k - \Sigma$ and desingularization $\tilde{F}_k$ of F_k on another side are more subtle: both involve the information on the type of singularities of C.

However in the important case when C is algebraic there is the following simple result.

LEMMA 2.6: Let C be an irreducible algebraic curve in CP^2. Then

$$H_1(F_k - \Sigma, Q) = H_1, Q)$$

For the proof we refer to [23] or [15]. (See also [11], p. 30.)

3. THE SECOND ALEXANDER MODULE

In this section we shall prove Theorem 2. Let R be either Z/pZ or Q. We first show that $A_2(R)$ is the free $R[t,t^{-1}]$ module of the rank $b = \mathrm{rk}\, H_1(C) + d - 1$. Let us consider the map $p: F_k \to CP^3$ which is the composition of the projection on the projective closure of the surface defined by $z^k = f(x,y)$ and the identical imbedding. Let H be the plane defined by $z = 0$. Then CUL is $p^{-1}(H)$. Let U be the regular neighbourhood of H. Then by [7] Theorem 2.1 we have $\pi_1(CUL) \twoheadrightarrow \pi_1(F_k)$. (Recall that F_k is normal and hence locally irreducible.) Therefore $H_1(CUL,R) \twoheadrightarrow H_1(F_k,R)$.

Now from the exact sequence

$$H_1(CUL,R) \to H_1(F_k,R) \to H_1(F_k,CUL,R) \to 0$$

we obtain that $H_1(F_k,CUL,R) = 0$. By Poincare duality

$$0 = H_1(F_k,CUL,R) = H^3(F_k - CUL,R) = H_3(F_k - CUL,R).$$

Thus Lemma 2.4 implies that the numbers c_k^2 are zeroes for any k. In other words the Alexander polynomial $\Delta^2(R)$ of the torsion part of $A_2(R)$ is relatively prime to $t^k - 1$ for any k.

On the other hand by Lemma 2.3 and Theorem 1, $\Delta^2(R)$ is cyclotomic.

This implies that in fact $\Delta^2(R) = 1$. Therefore $A_2(R)$ is a free $R[t,t^{-1}]$-module. The claim about the rank of $A_2(R)$ is contained in Lemma 2.

In order to conclude the proof of Theorem 2 we shall use the theorem of H. Bass [1] which shows that $A_2(Z) = Z[t,t^{-1}] \otimes P_0$ where P_0 is a projective Z-module, i.e., free. Triviality of $A_3(Z)$ was already shown in Section 2.

REMARK: Professor D. Summers showed to me that for any finite 2-dimensional complex (with H_1 isomorphic to Z) and any field R, $A_2(R)$ is free $R[t,t^{-1}]$-module.

This gives an alternative argument for the proof of the Theorem 2 because $CP^2 - CUL$ has homotopy type of 2-dimensional complex as 2-dimensional Stein manifold (cf. Section 5, Remark 1).

4. TORSION OF THE BRANCHED COVERINGS OF CP^2

In this section we shall prove Theorem 3. Let X be a connected complex with $H_1(X,Z) = Z$. Let $\tilde{X}$ and X_k be infinite and k-fold cyclic coverings of X respectively. The exact sequence of chain complexes (cf. 2.11)

$$0 \to C(\tilde{X}) \xrightarrow{t^k - 1} C(\tilde{X}) \to C(X_k) \to 0$$

gives

$$H_1(\tilde{X},Z) \xrightarrow{t^k - 1} H_1(X,Z) \to H_1(\tilde{X}_k) \to Z \to 0. \qquad (4.1)$$

What is essentially proven in [22] is that the order of $\mathrm{Coker}(t^k - 1)$ is equal to $\Pi_{i=1}^{k-1} \Delta_X(\omega^i) \cdot |T_k|$ where Δ_X is the Alexander polynomial of X (cf. Section 2, [15]) and $|T_k|$ is the order of Coker of $t^k - 1$ restricted on the subgroup of Z-torsion elements of $H_1(\tilde{X},Z)$.

Now (4.1) and Lemma 5 show that $\mathrm{Coker}(t^k - 1)$ is isomorphic to $H_1(F_k - \Sigma, Z)$ which gives the first part of Theorem 3.

To show the second part we observe that the projective closure $\bar{F}_k$ of the surface given by $z^k = f(x,y)$ is simply connected, as in any hypersurface in CP^3 ([7], Cor. 5.3).

Now $p: F_k \to \bar{F}_k$ is one-to-one everywhere except for the points of F_k which map p takes onto line L in infinity ([15]). The Van Kampen theorem applied to $F_k - \tilde{L} = \bar{F}_k - L$ and to the regular neighborhood of L in F_k and $\bar{F}_k$ respectively implies that $\ker(\pi_1(F_k) \to \pi_1(\bar{F}_k))$ is generated by the loop in the preimage of L in F_k connecting the branching points of $\tilde{L} \to L$. If α and β be paths in L and C respectively connecting two branching points and δ is a disk in CP^2 bounding $\alpha\beta^{-1}$, then the union of two of k preimages of δ under the projection $F_k \to CP^2$ gives the disk in F_k with the boundary coinciding with the union of two paths projecting on β. Therefore F_k is simply connected.

Now let E_{p_i} be an exceptional set of a desingularization $\tilde{F}_k \to F_k$ corresponding to a singular point p_i of the curve C. If t^k-1 and the local Alexander polynomial of the singularity p_i are relatively prime then $\pi_1(E_{p_i}) = 0$.

Indeed if $\pi_1(E_{p_i}) \neq 0$ then $H_1(E_{p_i})$ should have a free summand because E_{p_i} is equivalent to a bouquet of Riemann surfaces and circles ([12]). Hence the first Betti number of the boundary $\partial T(E_{p_i})$ of the tubular neighborhood of E_{p_i} in $\tilde{F}_k$ is non-zero because $H_1(\partial T(E_{p_i}))$ maps onto $H_1(E_{p_i})$. (Recall that $H_1(T(E_{p_i}), \partial T(E_{p_i})) = H^3(E_{p_i}) = 0$.

But $\partial T(E_{p_i})$ is the k-fold cyclic branched cover of S^3 branched over the link of singularity of C, which has one component in unibranched case. If the Alexander polynomial of the knot is relatively prime to t^k-1 then the first homology group of the k-fold cyclic branched covering is a torsion group. Therefore under our assumptions $\pi_1(E_{p_i}) = 0$.

By Van Kampen theorem $\pi_1(\tilde{F}_k)$ can be obtained as a series of amalgamations of $\pi_1(F_k-\Sigma)$ with $\pi_1(E_{p_i})$ all of which are trivial. Hence

$$\pi_1(F_k) = \pi_1(F_k - \Sigma)/\{\pi_1(\partial T(E_{p_i}))\}$$

where $\{\ \}$ denotes the normal subgroup generated by $\pi_1(\partial T(E_{p_i}))$ for all i. But $\pi_1(F_k)$ is given by the same formula. Hence the Theorem 3 is proven.

REMARK: The proof given above contains the following result.

THEOREM 4.1: The singularity $z^k = f(x,y)$ where f has isolated singularity at the origin is quasirational (i.e., desingularization contains only rational curves and does not have cycles) only if the Alexander polynomial of f and t^k-1 have no common roots besides $t = 1$.

The following fact was kindly pointed out to me by Professor S. Abhyankar. The singularities $z^k = f(x,y)$ are quasirational for any k if and only if f has only nodes as singularities. This follows from 4.1. Indeed the Alexander polynomial of algebraic singularity f has only root $t = 1$ if and only if f has at worse a node.

EXAMPLES AND CONCLUDING REMARKS

In the examples below we compute the Alexander modules in the cases when the fundamental group is known. Let

$$\pi_1(CP^2 - CUL) = G, \quad G' = [G,G] \quad \text{and} \quad G'' = [G',G'].$$

Note that $A_1(Z)$ can be identified with G'/G'' considered as a $Z[t,t^{-1}]$-module where the action of t on G'/G'' is obtained from the extension

$$0 \to G'/G'' \to G/G'' \to Z \to 0$$

(see [19]).

EXAMPLE 5.1: Let C be a quartic with 3 cusps. Then

$$A_1(Z/3Z) = (Z/3Z)[t,t^{-1}]/t+1 \quad \text{and}$$
$$A_1(Z/pZ) = 0 \quad \text{for any} \quad p \neq 3. \tag{5.1}$$

Indeed $\pi_1(CP^2-C)$ is the metacyclic group of the order 12 ([26]). This implies

$$1 \to Z/3Z \xrightarrow{i} \pi_1(CP^2 - CUL)) \xrightarrow{j} Z \to 1 \tag{5.2}$$

where j is the abelianization map.

Clearly the action of the generator of Z on $Z/3Z$ is non-trivial. This implies (5.1).

EXAMPLE 5.2: Let C_3 and $C_{2,2}$ be the curves constructed in [5]. Recall that C_3 and $C_{2,2}$ are the curves of degree 12 with 12 cusps and the number of nodes of C_3 and $C_{2,2}$ is equal to 21 and 22 respectively. Their fundamental groups are given as the following semi-direct products

$$1 \to K_{27} \to \pi_1(CP^2 - C_3) \to SL_2(Z) \to 1 \qquad (5.3)$$

and

$$1 \to Q_{64} \to \pi_1(CP^2 - C_{2,2}) \to SL_2(Z) \quad j \qquad (5.4)$$

where K_{27} (resp. Q_{64}) is the unique non-abilian group of order 27 (resp. 64) and exponent 3 (resp. 4). In this case we have

$$A_1(Z) = Z[t,t^{-1}]/t^2 - t + 1. \qquad (5.5)$$

Let us consider C_3. Note that the commutator subgroups of $\pi_1(CP^2 - CUL)$ and $\pi_1(CP^2 - C)$ are isomorphic for any irreducible curve as follows from the snake lemma applied to the diagram

$$\begin{array}{ccccccccc} 0 & \to & Z & \to & \pi_1(CP^2\text{-}CUL)) & \to & \pi_1(CP^2 - C) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & Z & \to & Z & \to & Z/d & \to & 1 \end{array} \qquad (5.6)$$

(vertical arrows are the abelianization homomorphisms, d is the degree of C and d the horizontal row follows from [24]). We shall show that

(a) $\pi_1(CP^2 - C)'/\pi_1(CP^2 - C)''$ is isomorphic to $Z \oplus Z$.

(b) There is $Z[t,t^{-1}]$ homomorphism of this module onto the Alexander module of the braid group B_3. This clearly implies (5.5).

The sequence 5.3 implies

$$1 \to K_{27} \to \pi_1(CP^2 - C_3)' \to F_2 \to 1 \qquad (5.7)$$

where F_2 is the free group on two generators. Indeed both groups $SL_2(Z)/SL_2(Z)'$ and $\pi_1(CP^2\text{-}C_3)/\pi_1(CP^2\text{-}C_3)'$ are isomorphic to $Z/12Z$. F_2 can be identified with the subgroup of $SL_2(Z)$ generated by

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Now K_{27}/K_{27}' is isomorphic to $Z_3 \oplus Z_3$ and the action of F_2 on $Z_3 \oplus Z_3 - \{0\}$ is transitive. Hence F_2 acts on $K_{27}/K_{27} \cap \pi_1(CP^2\text{-}CUL)'' - \{0\}$ transitively. Hence $\pi_1(CP^2\text{-}C_3)'/\pi_1(CP^2\text{-}C_3)$ is isomorphic to $Z \oplus Z$.

The part (b) follows from the fact that the central extension of $SL_2(Z)$ corresponding to the extension (5.6) is B_3, i.e., $\pi_1(CP^2 - CUL))$ maps onto B_3.

<u>EXAMPLE 3</u>: Let P_n be the branching curve of the generic projection of a non-singular surface in CP^3. Then, as shown by B. Moishezon ([18]) $\pi_1(CP^2\text{-}CUL))$ is B_n. The Redemeister-Shreier process leads to the explicit computation of B_n'/B_n'' (cf. [10]) and we get for the curve P_n

$$\begin{array}{lll} A_1(Z) = Z[t,t^{-1}]/t^2 - t + 1 & \text{for} & n = 3, 4 \\ A_1(Z) = 0 & & n > 4 \end{array} \qquad (5.7)$$

The formulas (5.7) also are true for the curves dual to the rational nodal curves of degree n for which the fundamental groups are the braid groups of a sphere with n-strings (cf. [5], [26]).

<u>EXAMPLE 4</u>: Let $C_{p,q}$ be Oka's curve ([20])

$$(X^p + Y^p)^q + (Y^q + Z^q)^p = 0 \qquad (5.8)$$

(p,q relatively prime).

Then

$$A_1(Z) = Z[t,t^{-1}]/\frac{(t-1)(t^{pq}-1)}{(t^p-1)(t^q-1)}$$

because $\pi_1(CP^2 - (CUL))$ is isomorphic to the group of the torus knot of type (p,q).

<u>REMARK</u>: Theorem 2 can be checked directly for the nodal curves. Indeed (cf. [4], [6], [7]) $\pi_1(CP^2\text{-}CUL) = Z/d$. Hence $\pi_1(CP^2\text{-}(CUL)) = Z$.

On the other hand $CP^2 - (CUL)$ has the homotopy type of a 2-dimensional complex since it is a 2-dimensional Stein manifold ([9]). According to [3] $CP^2\text{-}CUL$ has the homotopy type of the wedge $S^1 \vee S^2 \vee \ldots \vee S^2$ which implies Theorem 2 in this case.

REMARK 2: The outstanding problem is to characterize the groups of algebraic curves $\pi_1(CP^2-(CUL))$. Note that these groups are not in general knot groups which were characterized by Kervaire [13]. $\pi_1(CP^2-CUL)$ admits a system of conjugate generators and has H_1 isomorphic to Z, but H_2 is not trivial in general. Indeed for the braid groups (Example 3) one has $H_2(B(n),Z) = Z_2$ according to Arnold ([2]) at least for $n \geq 8$.

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