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A CHARACTERIZATION OF PLANE CURVE SINGULARITIES WITH ONE CHARACTERISTIC PAIR

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In paper [1], on the resolution of certain types of surface singularities, S. Abhyankar suggested the following characterization of the singularities of plane curves with one characteristic pair.

An *irreducible analytic curve* $F(X, Y) = 0$ *has one characteristic pair if and only if there are non-units* $A(T, U)$ *and* $B(T, U)$ *in* $k[[T, U]]$ *which are devoid of a non-unit common factor such that*

$$(1) \quad F(A(T, U), B(T, U)) = \prod_{r=1}^{s} [U - F_r T + \sum_{i+j>1} F_{rij} T^i U^j]^{a(r)}$$

where $a(r)$ *are positive integers and where* F_r *and* F_{rij} *are elements in* k *such that* $F_1, \ldots, F_s$ *are pairwise distinct.*

The purpose of this note is to prove this characterization in the case $k = \mathbf{C}$.

It is well known that a singularity has one characteristic pair if and only if the link of this singularity is a torus knot. On the other hand, according to Burde and Ziechang ([2]), torus knots are the only knots whose groups have a non-trivial center. We show that the existence of a factorization (1) implies non-triviality of the center of the group of the link of singularity F.

First, observe that the fundamental group of the link of the singularity $\tilde{F}(T, U)$ at the right hand side of (1) has as the center an infinite cyclic group. Indeed this singularity is equivalent to $\Pi_{r=1}^{s} (U - F_r T)^{a(r)}$. The complement to its link is fibred over $S^2 -$ [s points] with fibre S^1 which represents the generator of the center. (Here and in what follows, S^n denotes a sphere of dimension n). The map is just given by $(T, U) \to (T : U)$. In fact it is easy to show that the fundamental group of this link has a presentation

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with generators $a_1, \ldots, a_s$ and relations $a_1a_2 \cdots a_s = a_2 \cdots a_sa_1 = a_sa_1 \cdots a_{s-1}$. The center is the subgroup generated by $a_1 \cdots a_s$.

Now

$$X = A(T, U)$$
$$Y = B(T, U) \tag{2}$$

defines the map $\alpha : S^3 - K_{\tilde{F}} \to S^3 - K_F$ where S^3 on the left and on the right of the arrow denotes the boundary of a small ball centered at the origin in the (T, U) and (X, Y) planes respectively, $K_{\tilde{F}}$ and K_F are the links of singularities $\tilde{F}$ and F.

The image of the center of $\pi_1(S^3 - K_{\tilde{F}})$ in $\pi_1(S^3 - K_F)$ is non-trivial because it has a non-trivial image in $H_1(S^3 - K_F, Z)$. Indeed the loop representing the generator of the center of $\pi_1(S^3 - K_{\tilde{F}})$ is homologous to the union of small circles simply linked with each component of $K_{\tilde{F}}$.

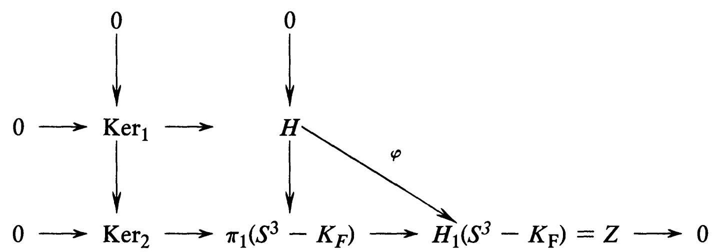
Moreover, $\pi_*(\pi_1(S^3 - K_{\tilde{F}}))$ has a finite index in $\pi_1(S^3 - K_F)$. This follows, for example from the diagram

$$\begin{array}{ccccc}
\pi_1(S^3 - K_{\tilde{F}} - B_{\tilde{F}}) & \longrightarrow & \pi_1(S^3 - K_{\tilde{F}}) & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \\
\pi_1(S^3 - K_F - B_F) & \longrightarrow & \pi_1(S^3 - K_F) & \longrightarrow & 1
\end{array}$$

where $B_{\tilde{F}}$ and B_F are the branching sets of map (2) and from the fact that the left arrow is induced by an unbranched covering of finite degree (not exceeding $\deg A \cdot \deg B$).

This implies that there is a normal subgroup of $\pi_1(S^3 - K_F)$ with a non-trivial center. Indeed if g_i's are representatives of cosets of $\alpha_*\pi_1(S^3 - K_{\tilde{F}})$ in $\pi_1(S^3 - K_F)$ then $H = \bigcap_i g_i^{-1}\alpha_*(\pi_1(S^3 - K_F))g_i$ can be used. H is a subgroup of finite index in $\alpha_*(\pi_1(S^3 - K_{\tilde{F}}))$ and hence H contains a central element of $\alpha_*(\pi_1(S^3 - K_{\tilde{F}}))$.

Note that the center of H ought to be an infinite cyclic group because H is an extension of a free group by Z as follows from the diagram



and from the fact that Ker_2 is a free group. On the other hand it is easy to see that such an extension can have only cyclic center (see e.g. [2], p. 170).

If c is a central element of H and g is any element of $H_1(S^3 - K_F)$ then $g^{-1}cg$ is a central element of H, and because the center of H injectively maps in $H_1(S^3 - K_F, Z)$ it cannot be anything else but c. i.e., c is a central element of $\pi_1(S^3 - K_F)$. As mentioned above [2] implies that K_F is a torus knot.

Finally, I would like to thank Prof. S. Abhyankar for useful discussion and introducing quasirational singularities to me.

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