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Remarks on moduli spaces of complete intersections

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1. Introduction.

In this paper we give an algorithm to compute the dimensions of the space of moduli of a complete intersection. This extends the treatment of hypersurfaces given by Kodaira and Spencer [8, §14]. One consequence, explained in §4, is that we can find diffeomorphic complete intersections of dimension three (and homeomorphic ones of dimension two) which lie in different dimensional components of the moduli space.

Our starting point is the results of E. Sernesi who showed [14] that the family $\mathcal{V}$ of complete intersections of fixed multidegree d whose defining polynomials have coefficients close to those of a given variety V_n, $n \geq 2$, is a complete complex analytic family of deformations of V except in the case of K-3 surfaces: $n = 2$ and $d = (4), (3,2)$, or $(2,2,2)$. It follows that any sufficiently small deformation of V is again a global complete intersection of the same multidegree. We let

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$V_n(d_1,\ldots,d_r)$, or simply V_n, denote a complete intersection of codimension r in P_{n+r} with multidegree $d = (d_1,\ldots,d_r)$ and $d_i \geq 2$.

Proposition 1 (Sernesi). $H^1(V_n, T_{P_{n+r}}|V_n) = 0$ for $n \geq 2$ except for the K-3 surfaces.

Sernesi used this in proving that the space of infinitesimal deformations of V is all of $H^1(V,T_V)$. It follows that the dimension, denoted m(V), of any effective, complete family of deformations is dim $H^1(V,T_V)$, [8, §§6 and 11].

To compute $H^1(V,T_V)$ we use the exact cohomology sequence corresponding to the exact sequence of sheaves [3, p.182]

(1) $$0 \longrightarrow T_{V_n} \longrightarrow T_{P_{n+r}}|V \longrightarrow N \longrightarrow 0$$

where N is the normal sheaf of V_n in P_{n+r}. In §3 we prove

Proposition 2. For any complete intersection except for the quadratic hypersurface

$$H^0(V,T_V) = 0 \qquad d \neq (2).$$

In general if V is a compact complex analytic manifold and $H^0(V,T_V) = 0$, the Kuranishi space is a local space of moduli for V. In our case, by Sernesi's result, its dimension is m(V) = dim $H^1(V,T_V)$, cf. [16, Chapter 2], [17, Theorem 1.1]. It also follows from $H^0(V,T_V) = 0$ that V admits no continuous group of analytic automorphisms. Under the stronger assumption that V has ample canonical bundle, Kobayashi showed [9] that Aut V is finite and Narasimhan and Simha [12] showed the existence of a global moduli space. They show the set of isomorphism classes of complex structures on V with ample canonical bundle has a natural structure as a Hausdorff complex space given in a neighborhood of a particular structure V_t by the Kuranishi space of V_t modulo Aut V_t.

As a consequence of Proposition 2 we can extend Kobayashi's result in the case of complete intersections.

Corollary. If V is a complete intersection of dimension ≥ 2 and not a K-3 surface or a quadratic hypersurface, then Aut V is finite.

Proof. Let $G = \text{Aut } P_{n+r} = PGL(n+r+1)$. The stabilizer $N_G(V) = \{g \in G \mid gV \subset V\}$ is an algebraic group, cf. [1, p.97]. The proof of Theorem 8.2 in [10, I p.479] shows that except for K-3 surfaces, Aut V = $N_G(V)$. But Proposition 2 implies that Aut V is zero dimensional for degree > 2. The Corollary follows.

For a K-3 surface Aut V may be infinite; in [15, p.288] Severi gave an example of a surface of degree 4 with infinite automorphism group. In [11] Matsumura and Monsky give a more algebraic proof, which holds also in nonzero characteristic, that Aut V is finite for hypersurfaces (with the same exceptions.) They also prove that for a generic hypersurface V, Aut V = 0.

In §2 we present a formula for m(V). The proof of Proposition 2 is given in §3. Finally §4 contains examples in dimensions 2 and 3 of complete intersections whose moduli spaces have components of different dimensions and a further survey of results on components.

2. Computing m(V).

We will need the following facts about the cohomology of line bundles on V. For $d = (d_1,\dots,d_r)$ and for any $s \le r$ define

$$q(\ell,n,s,d) = \dim H^0(V_n(d_1,\dots,d_s),\mathcal{O}_V(\ell)).$$

Lemma 1. Let V_n be a complete intersection of multidegree $d = (d_1,\dots,d_r)$ with $d_i \ge 2$.

(a) $H^i(V,\mathcal{O}(\ell)) = 0$ for $i \ne 0,n$.

(b) The function q is determined by the recurrence relation

$$q(\ell,n,r,d) = \begin{cases} 0 & \ell < 0 \\ \binom{n+\ell}{\ell} & r = 0 \\ q(\ell,n+1,r-1,d) - q(\ell-d_r,n+1,r-1,d) & r > 0 \end{cases}$$

(c) $$\Sigma_{\ell=0}^{\infty} q(\ell,n,r,d)t^{\ell} = (1-t)^{-n-r-1}\,\Pi_{i=1}^{r}(1-t^{d_i})$$

Proof. (a) is found for example as an exercise in [3, p.231]. (c) is proved in [13, p.131]. (b) follows from (c) and conversely. Alternatively (a) and (b) follow by induction on r from the case of projective space [3, p.225] using the exact cohomology sequence coming from the sequence [4, 16.2.1]

$$0 \longrightarrow \mathcal{O}_X(\ell-d_r) \longrightarrow \mathcal{O}_X(\ell) \longrightarrow \mathcal{O}_{X\cap W}(\ell) \longrightarrow 0$$

where $X = V_{n+1}(d_1,\dots,d_{r-1})$ and $W = V_{n+r-1}(d_r)$.

We have emphasized the recurrence relation (b) because it is useful for computation. There is another interpretation of the function q which may help to clarify its behavior. Writing the power series (c) as

$$(1+t+t^2+\dots)^{n+1}\,\Pi_{i=1}^{r}(1+t+\dots+t^{d_i-1}),$$

we see that the coefficient of t^{ℓ} is equal to the number of nonnegative integer solutions to the equation

$$x_1 + \ldots + x_{n+r+1} = \ell \text{ with } 0 \le x_i < d_i \text{ for } i = 1,\ldots,r.$$

This of course equals the number of integer points inside an obviously defined polyhedron.

The function $q(\ell,n,r,d)$ is symmetric in $(d_1,\ldots,d_r)$ but it is not polynomial and hence does not depend only on the elementary symmetric polynomials. On the other hand if

$$\ell > \Sigma(d_i - 1) - n - 1,$$

then by the Kodaira vanishing theorem [4, 18.2.2]

$$q(\ell,n,r,d) = \chi(V,\mathcal{O}(\ell))$$

which by the Riemann-Roch formula is a polynomial in $\ell + \frac{1}{2}c_1(V)$ and the Pontryagin classes of V [4, p.150] and hence is polynomial in ℓ, the total degree, $\Pi\, d_i$, and the first n elementary symmetric functions of $(d_1,\ldots,d_r)$.

Theorem. If V_n is a complete intersection of multidegree $(d_1,\ldots,d_r)$ with $d_i \ge 2$ and with $n \ge 2$ and not a K-3 surface or a quadratic hypersurface, then the number of moduli

$$m(V) = 1 - (n + r + 1)^2 + \Sigma_{i=1}^{r} q(d_i,n,r,d).$$

Proof. We first find dim $H^0(V,T_{P_{n+r}}|V)$. Tensoring the exact sequence [3, p.182]

$$0 \longrightarrow \mathcal{O}_{P_{n+r}} \longrightarrow \mathcal{O}_{P_{n+r}}(1)^{n+r+1} \longrightarrow T_{P_{n+r}} \longrightarrow 0$$

with $\mathcal{O}_V$, since T_P is flat we obtain

$$0 \longrightarrow \mathcal{O}_V \longrightarrow \mathcal{O}_V(1)^{n+r+1} \longrightarrow T_{P_{n+r}}|V \longrightarrow 0$$

The corresponding cohomology sequence gives

$$0 \longrightarrow H^0(V,\mathcal{O}_V) \longrightarrow H^0(V,\mathcal{O}_V(1))^{n+r+1} \longrightarrow H^0(V,T_{P_{n+r}}|V) \longrightarrow 0$$

Because V is not contained in any proper linear subspace of P_{n+r}, we have $\dim H^0(V, O_V(1)) = n + r + 1$, in agreement with Lemma 1. Hence $\dim H^0(V, TP_{n+r}|V) = (n+r+1)^2 - 1$.

In view of Propositions 1 and 2 the cohomology sequence corresponding to the sequence (1) becomes

$$(2) \quad 0 \longrightarrow H^0(V, TP_{n+r}|V) \longrightarrow H^0(V,N) \longrightarrow H^1(V,T_V) \longrightarrow 0$$

Since the normal sheaf N is isomorphic to

$$O_V(d_1) + \ldots + O_V(d_r),$$

the theorem follows from Lemma 1.

A table of values of $m(V)$ in cases of low degree is given at the end of §4.

For a hypersurface of degree d we have

$$q(d,n,1,d) = \binom{n+1+d}{d} - 1$$

by Lemma 1 and so we obtain the formula

$$m(V) = \binom{n+d+1}{d} - (n+2)^2 \quad \text{for } d > 2$$

of Kodaira and Spencer [8, (14.10)].

A quadratic hypersurface V is rigid, $H^1(V,T_V) = 0$ [8, p.406], and our formula gives instead

$$\dim H^0(V,T_V) = \tfrac{1}{2}(n+1)(n+2).$$

For the K-3 surfaces the formula gives 19, the dimension of the image of δ^* in (2).

As the codimension r increases a closed form expression for $m(V)$ becomes more complicated. For example if $r = 2$

$$m(V) = -1 - (n+3)^2 + \binom{n+2+d_1}{d_1} + \binom{n+2+d_2}{d_2} - \binom{n+2+d_2-d_1}{d_2-d_1} - \delta^{d_1}_{d_2}$$

assuming $d_2 \geq d_1$. Here δ is the Kronecker function.

A vanishing lemma.

In this section we compute $H^0(V,T_V)$. By Serre duality this group is isomorphic to $H^n(V,\Omega_V^1(\Sigma(d_i-1) - n - 1))$. If V has ample canonical bundle, $\Sigma(d_i-1) - n -1 > 0$ and this group vanishes by a criterion of Akizuki and Nakano [8, (11.12)]. We reduce all but one of the remaining cases to their result using a theorem of Bott and an idea of Kodaira and Spencer [8, Lemma 14.2].

Lemma 2. Let V_n be a complete intersection. Then

$$H^q(V,\Omega_V^p(k)) = 0 \text{ for } p+q \geq n+1 \text{ and } k > p-q.$$

Proof. For k > 0 this is the result of Akizuki and Nakano. The proof is by induction on the codimension r of V. For r = 0, V = P_n and our lemma follows from Bott's result [8, p.405]. Note that we may assume $q \geq 1$ since $\Omega^{n+1} = 0$.

For the inductive step assume V_n is a hypersurface of degree j in the complete intersection W_{n+1} of codimension r-1. The pair of exact sequences

$$0 \longrightarrow \Omega_W''^p \longrightarrow \Omega_W^p \longrightarrow \Omega_V^p \longrightarrow 0$$

$$0 \longrightarrow \Omega_W^p \longrightarrow \Omega_W''^p(j) \longrightarrow \Omega_V^{p-1} \longrightarrow 0$$

of Kodaira and Spencer [7, (3) and (4)] tensored with $\mathcal{O}_W(k+j)$ and $\mathcal{O}_W(k)$ yield a pair of exact sequences from whose cohomology sequences we take two short sections:

$$H^{q-1}(V,\Omega_V^{p+1}(k+j)) \longrightarrow H^q(W,\Omega_W''^{p+1}(k+j)) \longrightarrow H^q(W,\Omega_W^{p+1}(k+j))$$

$$H^q(W,\Omega_W''^{p+1}(k+j)) \longrightarrow H^q(V,\Omega_V^p(k)) \longrightarrow H^{q+1}(W,\Omega_W^{p+1}(k)).$$

Since dim W = n+1 and $j \geq 2$, the hypotheses are satisfied by the first and last groups in the first sequence and the last group in the second sequence. Since W has lower codimension, the last groups are zero by induction. For large k (so $k+j > 0$) the first group in the first sequence is zero by the criterion of Akizuki and Nakano. The lemma follows by induction on $-k$.

Our lemma implies Proposition 2 provided

$$\Sigma(d_i-1) - n - 1 > 1 - n$$

hence for all cases except $d = (2)$, (3) and, (2,2). The case $d =$ (3) uses a more complete form of Bott's theorem with $W = P_{n+1}$. It is covered by [8, Lemma 14.2].

In the case of a complete intersection of two quadrics the vanishing results above are not sufficient. One can use the following alternative argument. We have a commutative diagram

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ & & & & H^0(V,T_V) & & \\ & & & & \downarrow & & \\ 0 \longrightarrow H^0(V,O_V) & \longrightarrow & H^0(V,O_V(1))^{n+r+1} & \longrightarrow & H^0(V,T_{P_{n+r}}|V) & \longrightarrow & 0 \\ & & \downarrow \mu & & \downarrow & & \\ & & H^0(V,O_V(d_1) + \ldots + O_V(d_r)) & = & H^0(V,N) & & \end{array}$$

The rightmost 0 comes from $H^1(V,O_V) = 0$ for $n \geq 2$ and the map μ is given by

$$(R_0,\ldots,R_{n+r}) \longrightarrow (\Sigma_{i=0}^{n+r} R_i \frac{\partial F_1}{\partial x_i},\ldots, \Sigma_{i=0}^{n+r} R_i \frac{\partial F_r}{\partial x_i})$$

where

$R_i = \Sigma_{i=0}^{n+r} r_{ij}x_j$ are linear forms and $F_1 = 0,\ldots, F_r = 0$ are the defining equations of $V_n(d_1,\ldots,d_r)$. The kernel of μ is determined by the conditions

$$\Sigma_{i=0}^{n+r} R_i \frac{\partial F_k}{\partial x_i} = \Sigma_{\ell=1}^{r} a_{k\ell}F_\ell \text{ for } k = 1,\ldots,r.$$

For $d = (2,2)$ we take

$$F_1 = x_0^2 + \dots + x_{n+2}^2$$
$$F_2 = c_0x_0^2 + \dots + c_{n+2}x_{n+2}^2$$

with $c_i \neq c_j$ for $i \neq j$. The equations imply $r_{ij} = 0$ for $i \neq j$ and $r_{ii} = \frac{1}{2}a_{11}$ for $i = 0,\dots,n+2$. Hence $\dim \ker\mu = 1$. This implies $H^0(V,T_V) = 0$. This completes the proof of Proposition 2.

In the case of a quadric hypersurface the computation of $\ker\mu$ yields

$$\dim H^0(V,T_V) = \tfrac{1}{2}(n+1)(n+2)$$

in agreement with the remark in §2.

4. Examples and remarks.

In [5, 6] Horikawa gave examples which show that the moduli space of algebraic structures on a given smooth 4-manifold can have different components of different dimensions. In [10] the authors showed that in any complex dimension > 2 and for any positive integer k, there are k distinct complete intersections which are all diffeomorphic. In fact they have equivalent underlying almost complex structures. However they lie in distinct irreducible components of the moduli space of complex structures on the underlying smooth manifold. We obtained the same result in dimension two for structures on a homeomorphism type. Recent work of Catanese [2] provides for any k a homeomorphism type of real dimension 4 supporting complex structures lying in components of the moduli space of k different dimensions. It is natural to expect similar behavior among complete intersections.

For surfaces, the homotopy type is determined by the middle Betti number and the signature and type of the intersection pairing which can be computed from the total degree and the first two symmetric functions of $\vec{d}$. By the work of Mike Freedman the homeomorphism type of these manifolds is determined by the homotopy type. We find the varieties $V_2(6,6,6,2,2,2,2)$ and $V_2(8,4,4,3,3,3)$ are homeomorphic but have $m(V) = 7509$ and 9546 respectively.

There is a way to generate larger sets of homeomorphic surfaces from pairs. Suppose $\vec{d} = (d_1,\ldots,d_r)$ and $e = (e_1,\ldots,e_s)$ are two multidegrees with the same total degree and symmetric functions σ_1 and σ_2. Let $de = (d_1,\ldots,d_r,e_1,\ldots,e_s)$. Then the three multidegrees dd, de, and ee also have the same invariants σ_1, σ_2, and total degree. Similarly for ddd, dde, dee, and eee, etc. It is reasonable to expect that the corresponding $m(V)$'s will all be different. Unfortunately we have no general way to prove this.

In case $\vec{d} = (10,10,4,3,3)$ and $e = (12,6,5,5,2)$ which give homeomorphic surfaces we find

multidegree	$m(V_2)$
$\vec{d}$	23356
e	27005
dd	1695364
de	2234720
ee	2758226

Computation of the four cases which come from juxtaposing three terms and of the five cases which come from juxtaposing four terms gives four and five distinct moduli dimensions respectively.

For 3-folds the total degree, σ_1, σ_2, σ_3 determine the diffeomorphism type [10, I p.480]. In this case the varieties $V_3(14,14,5,4,4,4)$ and $V_3(16,10,7,7,2,2,2)$ are diffeomorphic with moduli dimensions 1028748 and 1191130 respectively. As above larger sets of diffeomorphic varieties can be generated. We have checked that the moduli dimensions are all different through the case of five distinct but diffeomorphic complete intersections.

These computations and the ones for the small table below were done with the aid of a computer using the recursive definition of the function q given in Lemma 1(b).

Table of m(V) for some complete intersections of low degree

multidegree	dim 2	dim 3
3	4	10
4	19 (K-3)	45
2,2	2	3
5	40	101
6	68	185
3,2	19 (K-3)	34
4,2	44	89
2,2,2	19 (K-3)	27
3,2,2	46	73

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