On the homotopy type of the complement to plane algebraic curves

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0. Introduction

Let C be an algebraic curve in $\mathbb{C}^2$ which may have arbitrary singularities. $\mathbb{C}^2 - C$ is a two-dimensional Stein manifold [AF] and therefore has the homotopy type of a two-dimensional complex. An algorithm for finding the fundamental group of $\mathbb{C}^2 - C$ was given by van Kampen in 1932 [VK]. In this note we propose an approach to finding the homotopy type of $\mathbb{C}^2 - C$. Recall that to any presentation of a group G with generators $x_1, \ldots, x_n$ and relations $r_1, \ldots, r_k$ one associates a two-dimensional complex with one zero dimensional cell, n one dimensional cells corresponding to the generators $x_1, \ldots, x_n$ and k two-dimensional cells corresponding to the relations $r_1, \ldots, r_k$. We describe the homotopy type of $\mathbb{C}^2 - C$ by describing a presentation of the fundamental group of $\mathbb{C}^2 - C$ such that the associated two-dimensional complex is homotopy equivalent to $\mathbb{C}^2 - C$. Note that presentations given in usual formulations of the van Kampen theorem have the associated 2-complex with a bigger Euler characteristic than $\mathbb{C}^2 - C$ (cf. [VK], [Ch]). Note that the presentation which we describe is the one given by Moishezon [M] for curves with nodes and cusps. In the first section we give a proof of the apparently known fact that the Wirtinger and Artin presentations of the fundamental groups of the complement to a knot and closed braid respectively have associated two-dimensional complexes homotopy equivalent to their complements in S^3. In the second section we describe the required presentation of $\pi_1(\mathbb{C}^2 - C)$ in terms of the braid monodromy introduced by B. Moishezon [M] and prove that for this presentation the associated 2-complex is homotopy equivalent to $\mathbb{C}^2 - C$. In the final part we consider the change of homotopy type of the complement to the plane curves in degenerations and consider computation of the homotopy type using the results of this paper.

1. Presentations of knot groups

a. The Wirtinger presentation

Let L be an oriented link in 3-sphere S^3 which in this section we shall view as $\mathbb{R}^3$ compactified by one point. Let $\bar{L}$ be a projection of L onto a plane H which we can assume given by the equation $z = 0$. We shall assume that this projection is an immersion such that the image $\bar{L}$ has only double points as singularities. We can make an isotopy of L to a link $\bar{\bar{L}}$ which coincides with $\bar{L}$ everywhere except for a neighborhood of a finite set D of points of $\bar{L}$ which contains the set of double points. Moreover we can assume that

a) $\bar{\bar{L}}$ coincides near any double point of $\bar{L}$ with the union of two curves one of which is one of two branches of $\bar{L}$ and another running under H, (see figure 1a) and

b) $\bar{\bar{L}}$ coincides near points of D which are not double points on $\bar{L}$ with a curve running under H (see figure 1b). For sufficiently small ε the link $\bar{\bar{L}}$ is a union of N arcs in the half-space $A_\varepsilon = \{(x, y, z) | z > -\varepsilon\}$ connected by underpasses. We consider a free

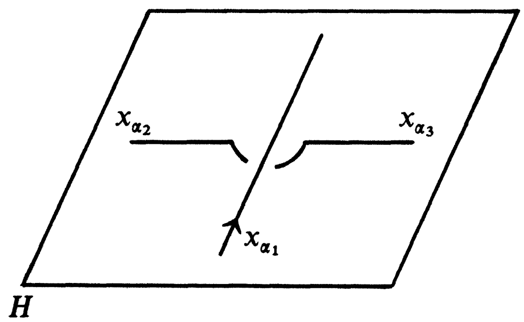

Figure 1a

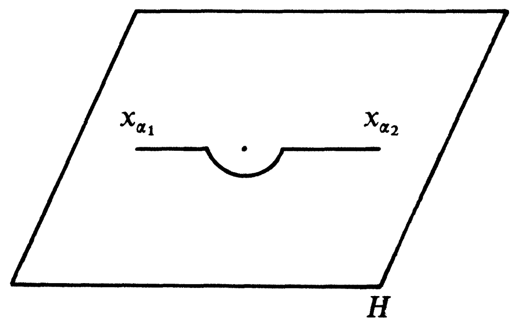

Figure 1b

group on N generators x_α, $\alpha = 1, \ldots, N$ corresponding to those arcs. Each point of D defines a relation of the following forms

a) $x_{\alpha_1} x_{\alpha_2} x_{\alpha_1}^{-1} x_{\alpha_3}^{-1} = 1$, or $x_{\alpha_1}^{-1} x_{\alpha_2} x_{\alpha_1} x_{\alpha_3}^{-1} = 1$ for each double point of $\bar{L}$ depending on whether the orientation of L induces left or right orientation at the double point,

b) $x_{\alpha_1} x_{\alpha_2}^{-1} = 1$ for each point of D which is not double.

It is well known (cf. [R]) that $\{x_1, \ldots, x_n | R_1, \ldots, R_{|D|-1}\}$ ($|D|$ is the number of points in D) is a presentation of $\pi_1(S^3 - L)$. It is called a Wirtinger presentation. Analysis of the proof of this fact leads to the following more precise result.

Lemma 1. *If $\bar{L} - D$ does not contain circles then the 2-complex associated to the Wirtinger presentation of $\pi_1(S^3 - L)$ described above is homotopy equivalent to $S^3 - L$.*

Proof. Let B_i denote an open box with the top face belonging to the plane $H_\varepsilon = \{(x, y, z) | z = -\varepsilon\}$ and such that the underpass corresponding to the i-th point from D belongs to B_i (see figure 2a). We shall assume that only in one box, say $B_{|D|}$, which corresponds to the $|D|$-th point for which the relation is dropped, the underpass does touch the bottom face, say at the point p. In all other boxes B_i the underpasses are in the interior of the B_i's. Let $C = S^3 - \left(A_\varepsilon \cup \bigcup_{i=1}^{|D|} B_i\right)$. Then $C - \bar{\bar{L}}$ is homeomorphic to a closed 3-ball with the point $\bar{\bar{L}} \cap \partial C$ omitted. Therefore we can retract $S^3 - \bar{\bar{L}}$ onto $\bar{A}_\varepsilon \cup \bigcup_{i=1}^{|D|} \bar{B}_i - \bar{\bar{L}}$ by retracting $C - \bar{\bar{L}} \cap \partial C$ onto $\partial C - L \cap \partial C$. Next each $\bar{B}_i - \bar{\bar{L}}$ ($i = 1, \ldots, |D| - 1$) can be retracted onto $\partial B_i - \bar{\bar{L}}$ which produces a retraction of

$\bar{A}_\varepsilon \cup \bigcup_{i=1}^{|D|} \bar{B}_i - \bar{L}$ onto $(\bar{A}_\varepsilon \cup \bar{B}_{|D|} - \bar{L}) \cup e_1 \cup \cdots \cup e_{|D|-1}$ where the e_i $(i=1,\dots,|D|-1)$ are 2-cells each of which is the union of the faces of B_i not belonging to the plane H_ε. Finally $(\bar{A}_\varepsilon \cup \bar{B}_{|D|}) - \bar{L}$ is homeomorphic to a ball from which we have removed an union of arcs the boundaries of which belong to the boundary of this ball. The latter is homeomorphic to a cylinder with $|D|$ vertical segments corresponding to overpasses of $\bar{L}$ removed (see fig. 2b). Hence $A_\varepsilon \cup B_{|D|} - \bar{L}$ can be retracted onto a wedge of circles $\alpha_1, \dots, \alpha_{|D|}$ corresponding to overpasses of $\bar{L}$. $\pi_1(A_\varepsilon \cup B_{|D|} - \bar{L})$ can be identified with the free group on the generators $x_1, \dots, x_{|D|}$ described above. Thus we obtain a retraction of $S^3 - \bar{L}$ onto $\alpha_1 \cup \cdots \cup \alpha_{|D|-1} \cup l_1 \cup \cdots \cup l_{|D|-1}$. Clearly the attaching maps of the cells e_i are as described earlier. Q.E.D.

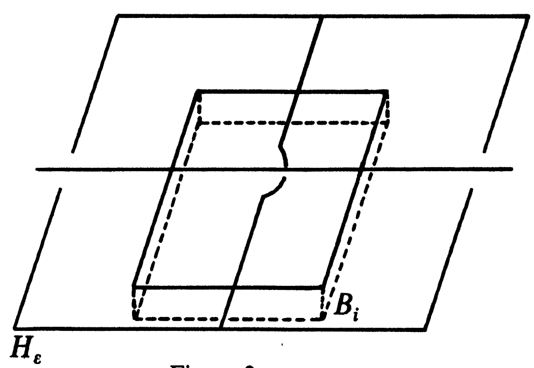

Figure 2a

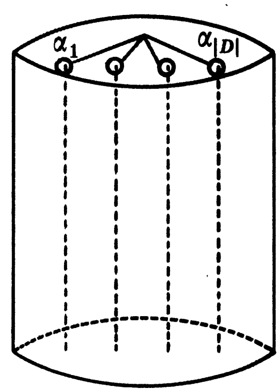

Figure 2b

b. The Artin presentation

Let β be a braid of n strings connecting sets S_1 and S_2 of n points in two disks D_1 and D_2 belonging to parallel planes P and $\bar{P}$. We can assume that D_1 and D_2 are the top and the bottom of a cylinder C which is the part of a torus T in S^3. By taking the union of β and n untwisted strings in $T-C$ one obtains the link $\hat{\beta}$ in S^3 which is the closed braid corresponding to β. Any link is isotopic to a closed braid (cf. [B], p. 42).

Recall also ([M]) that any braid can be viewed as an isotopy class of orientation preserving diffeomorphisms of D_1 which fix S_1 and induce the identity on ∂D_1. Each half twist of two strings corresponds to a rotation of a subdisk of D_1 by 180° about one of the segments of a chosen system of $(n-1)$ non-intersecting segments in D_1 connecting points in S_1. Any diffeomorphism of D_1 fixing the set S_1 and which is the identity on ∂D_1 is product of such half-twists which we denote in what follows by $X_1, \dots, X_{n-1}$. Any such diffeomorphism Φ induces an automorphism of $\pi_1(D_1 - S_1)$ which can be shown to be of the form

$$\Phi(y_i) = A_i y_{\mu(i)} A_i^{-1}$$

where the y_i are standard generators of $\pi_1(D_1 - S_1)$ (cf. [B] and see fig. 3a), μ is a permutation of n letters given by the braid β, and A_i are certain words in y_i, $i=1,\dots,n$.

Recall that one can take as the y_i's a system of simple loops each of which, assuming that S_1 is a set of points on horizontal diameter of D_1, can be described as the union of an arc, a vertical segment, and a small circle about a point of S_1 (see fig. 3a).

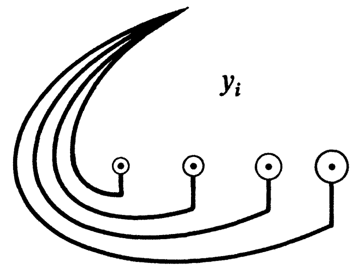

Figure 3a

It is well known (cf. [B]) that $\pi_1(S^3-\hat{\beta})$ has a presentation with generators $y_1, \ldots, y_n$ and relations

$$y_i = A_i y_{\mu(i)} A_i^{-1} \qquad i=1, \ldots, n-1.$$

We refer to this presentation of $\pi_1(S^3-\hat{\beta})$ as to the Artin presentation.

Lemma 2. *If a presentation of β as a reduced word in the X_i's contains all the generators $X_1, \ldots, X_{n-1}$ then the 2-complex associated with the Artin presentation is homotopy equivalent to $S^3-\hat{\beta}$.*

Proof. Let N denote the number of half twists in the braid β or equivalently the length of a reduced presentation of β as a word in generators $X_1, \ldots, X_{n-1}$. After an appropriate isotopy of β this number also will be equal to the number of double points of a projection of $\hat{\beta}$ into a plane K perpendicular to P. For any pair of points of $\hat{\beta}$ projecting into the same point of K, the point closest to K will be called an overpass and the farthest will be called an underpass (relative to the projections on K). The underpasses are naturally ordered by their distance from the plane P and we shall label them by numbers $1, \ldots, N$.

We shall define a series of presentations AW_k, $k=0, \ldots, N-1$ of the group $\pi_1(S^3-\hat{\beta})$ which is a mixture of Artin and Wirtinger presentations such that

1. AW_{k+1} can be obtained from AW_k by a sequence of Tietze transformations of the following type (cf. Section 3) (I): adding (or removing) a new generator g and a new relation r which expresses the added (removed) generator g as a word in other generators.

Clearly the two complexes associated with AW_{k+1} and AW_k have the same homotopy type.

2. AW_0 can be obtained from Wirtinger presentation by means of Tietze transformations of type (I).

3. AW_{N-1} can be obtained from the Artin presentation by means of Tietze transformations of type (I).

Let $y_1, \ldots, y_n$ be a system of generators chosen as above and with a base point B on the boundary of D_1. Let P_k be a plane parallel to P such that the part of β between P and P_k contains k half-twists. Let C_k be the part of the cylinder C between P and P_k. Let $\bar{y}_1^k, \ldots, \bar{y}_n^k$ be a system of generators of $\pi_1(C \cap P_k - \beta, B_k)$ (where B_k is a boundary point of $C \cap P_k$) which we can assume are projections of y_i's onto P_k. A choice of a path connecting B_k and B allows us to consider $\bar{y}_i^k, \ldots, \bar{y}_n^k$ as elements of $\pi_1(S^3 - \hat{\beta}_1, B)$.

Underpasses between P_k and $\bar{P}$ split the part of β between P_k and $\bar{P}$ into a union of arcs. Those arcs which do not intersect P_k correspond in one to one fashion with the underpasses between P_k and $\bar{P}$, by the correspondence relating to each underpass the arc having it as the top end. Each arc not intersecting P_k has as its end either two consecutive underpasses along this arc (e.g. α_{k+2}^k on fig. 3b) or an underpass and a point of $\beta \cap \bar{P}$ (e.g. α_{k+1}^k on fig. 3b). We denote these arcs by $\alpha_{k+1}^k, \ldots, \alpha_N^k$ where the lower index is the number of the underpass which is the top end of the arc. To each arc

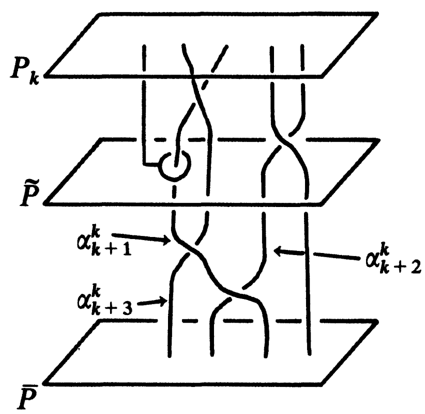

Figure 3b

α_s^k $(s = k+1, \ldots, N)$ we relate a loop in $S^3 - \hat{\beta}$ consisting of two parts. The first one is a curve on ∂C connecting B and a plane $\tilde{P}$ parallel to P and intersecting α_s^k between s-th and $(s+1)$-th underpass. The second part is a loop in the plane $\tilde{P}$ consisting of a union of an arc with a small circle about $\alpha_s^k \cap \tilde{P}$ chosen in such a way that the natural isotopy of $\tilde{P} - \beta$ into $P_s - \beta$ takes this composite loop to one of the $\bar{y}_i^s$'s. Denote these loops by $x_{k+1}^k, \ldots, x_N^k$. Let AW_k be the presentation with generators $y_1, \ldots, y_n$, $\bar{y}_1^k, \ldots, \bar{y}_n^k$, $x_{k+1}^k, \ldots. x_N^k$ and the following relations:

a) Artin relations: $\bar{y}_{\mu_k(i)}^k = \beta_k(y_i)$ $i = 1, \ldots, n-1$, where β_k is the part of β between P and P_k and μ_n is the permutation corresponding to β_k.

b) Matching relations: Let us consider an isotopy of $\bar{P} \cap C$ into $P \cap C$ (the bottom of C into the top) obtained by moving the disk $\bar{P} \cap C$ inside $T - C$. This gives an identification of $\pi_1(\bar{P} - \bar{P} \cap \hat{\beta})$ with $\pi_1(P - P \cap \hat{\beta})$ taking the set of standard generators $\bar{y}_1, \ldots, \bar{y}_n$ of $\pi_1(\bar{P} - \bar{P} \cap \hat{\beta})$ into $y_1, \ldots, y_n$. On the other hand each $\bar{y}_s (s = 1, \ldots, n)$ is identified by isotopy inside C with one of x_j^k's or y_l^k's. Combining these two identifications we get relations of the form $y_s = x_j^k$ (for each arc with lower end in $\bar{P}$) or $y_s = y_l^k$ (for each string of β connecting P_k and $\bar{P}$ and not containing underpasses). These relations, except for the one involving x_N^k, we call matching relations (there are $(n-1)$ of them).

c) Wirtinger relations corresponding to underpasses of $\hat{\beta}$ outside of C_k involving $\bar{y}_i^k$ and x_j^k.

Now in AW_{N-1} the matching relations have the form $\bar{y}_i^{N-1} = y_i$, $i \neq s$ where s is the lower end of α_N^{N-1} and by eliminating them one obtains the Artin presentation of $\pi_1(S^3 - \hat{\beta})$. This proves claim 3 above. To transform AW_k into AW_{k+1} one does the following:

a) Add new generators $\bar{y}_i^{k+1}$, x_j^{k+1}, $i = 1, \ldots, n$; $j = k+2, \ldots, N$

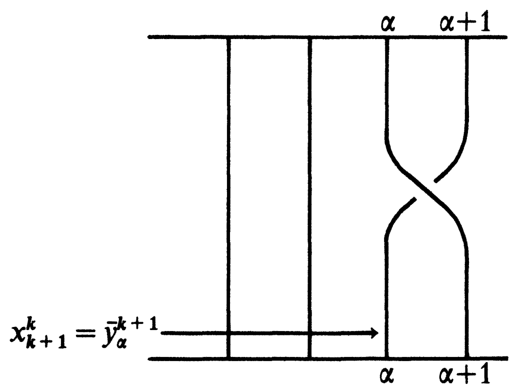

Figure 4

b) Add identifying relations $\bar{y}_i^k = \bar{y}_i^{k+1}$ for $i \neq \alpha$, $\alpha+1$ where α and $\alpha+1$ are indices of interchanged points between planes P_k and P_{k+1}; the relations $x_j^{k+1} = x_j^k$, $j = k+2, \ldots, N$; the relation $x_{k+1}^k = \bar{y}_\alpha^{k+1}$; and the relation $\bar{y}_{\alpha+1}^{k+1} = \bar{y}_\alpha^k$ in the case of right hand half twist (or $y_{\alpha+1}^k = y_\alpha^{k+1}$ and i.e. $y_{k+1}^k = y_{\alpha+1}^{k+1}$ in the case of left hand half twist).

c) Eliminate, using identifying relations from b), all $\bar{y}_i^k$ and $\bar{x}_j^k$ for $i \neq \alpha+1$ and eliminate $y_{\alpha+1}^k$ using the Wirtinger relation $\bar{y}_{\alpha+1}^k = \bar{y}_{\alpha+1}^{k+1} \bar{y}_\alpha^{k+1} (\bar{y}_{\alpha+1}^{k+1})^{-1}$.

After the substitutions, all Artin relations of the form $\bar{y}_{\mu_k(i)}^k = \beta_k(y_i^k)$ will change to $\bar{y}_{\mu(i)}^{k+1} = \beta_{k+1}(y_i^{k+1})$, so that one obtains the presentation AW_{k+1}^{k+1}.

Now in AW_0 one can eliminate all y_i but one, say y_α, by using matching relations and x_N^0 using Wirtinger relations. With the remaining generators $x_i, \bar{y}_\alpha$ $i = 1, \ldots, N$, one obtains a Wirtinger presentation. The assumption in lemma 2 implies the assumption in lemma 1. This concludes the proof of lemma 2.

Example. Let us consider the torus link of type (2. 4). The presentation AW_0 is given by

$$\{y_1, y_2, x_1^0, x_2^0, x_3^0, x_4^0 \mid y_1 y_2 y_1^{-1} = x_1^0, x_1^0 y_2 (x_1^0)^{-1} = x_2^0, x_2^0 x_1^0 (x_2^0)^{-1} = x_3^0, x_3^0 x_2^0 (x_3^0)^{-1} = x_4^0, x_3^0 = y_2\}$$

(cf. fig. 5a). The first four relations are Wirtinger relations and the last one is the matching relation. Elimination of y_2 using matching relations and x_2^0 using the last

Wirtinger relation gives the presentation

$$\{y_1, x_1^0, x_2^0, x_3^0 \mid y_1 x_3^0 y_1^{-1} = x_1^0,\ x_1^0 x_3^0 (x_3^0)^{-1} = x_2^0,\ x_2^0 x_1^0 (x_2^0)^{-1} = x_3^0\}$$

which is a Wirtinger presentation.

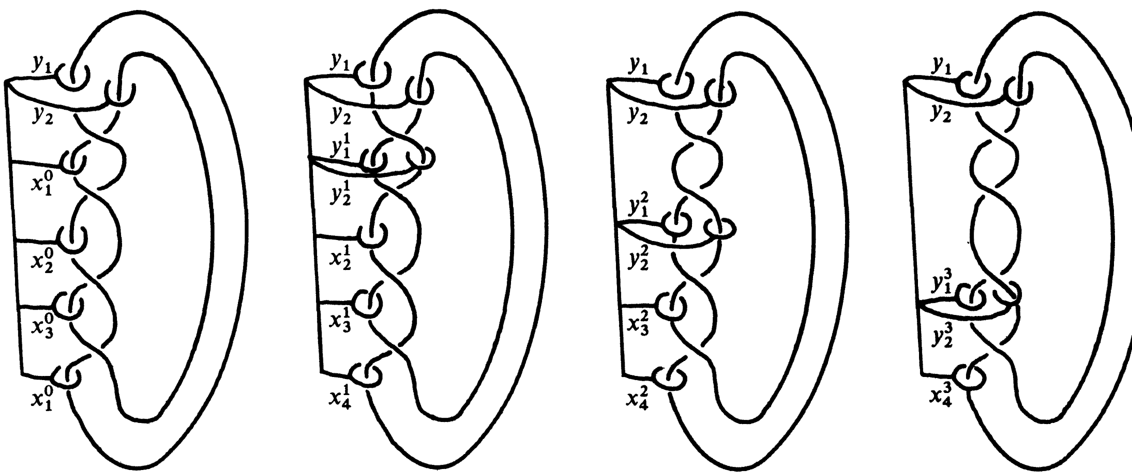

Figure 5a Figure 5b Figure 5c Figure 5d

The presentation AW_1 is given by

$$\{y_1, y_2, y_1^1, y_2^1, x_2^1, x_3^1, x_4^1 \mid y_2^1 = y_1,\ y_1^1 = y_1 y_2 y_1^{-1},\ y_1^1 y_2^1 (y_1^1)^{-1} = x_2^1,\ x_2^1 y_1^1 (x_2^1)^{-1} = x_3^1,\ x_3^1 x_2^1 (x_3^1)^{-1} = x_4^1,\ x_3^1 = y_2\}$$

(cf. fig 5b). To obtain AW_1 from AW_0 we rename the generators $x_1^0 \to x_2^1$, $x_3^0 \to x_3^1$, $x_4^0 \to x_4^1$, $x_1^0 \to y_1^1$, add a new generator y_2^1, and a new relation $y_2^1 = y_1$.

The presentation AW_2 is given by

$$\{y_1, y_2, y_1^2, y_2^2, x_3^2, x_4^2 \mid y_1 y_2 y_1 y_2^{-1} y_1^{-1} = y_1^2,\ y_1 y_2 y_1 = y_2^2,\ y_1^2 y_2^2 (y_1^2)^{-1} = x_3^2,\ x_3^2 y_1^2 (x_3^2)^{-1} = x_4^2,\ x_3^2 = y_2\}.$$

To obtain AW_2 from AW_1 we add new generators $y_1^2, y_2^2, x_3^2, x_4^2$ and relations $y_2^2 = y_1^1$, $x_2^1 = y_1^2$, $x_3^1 = x_3^2$, $x_4^1 = x_4^2$. Using these relations we eliminate $y_1^1, x_2^1, x_3^1, x_4^1$. The resulting presentation is

$$\{y_1, y_2, y_2^1 \cdot y_1^2, y_2^2, x_3^2, x_4^2 \mid y_2^1 = y_1,\ y_2^2 = y_1 y_2 y_1^{-1},\ y_2^2 y_2^1 (y_2^2)^{-1} = y_1^2,\ y_1^2 y_2^2 (y_1^2)^{-1} = x_3^2,\ x_3^2 y_1^2 (x_3^2)^{-1} = x_4^2,\ x_3^2 = y_2\}.$$

Finally we eliminate y_2^1 using the first relation in this presentation and rewrite the third relation using the second one. The presentation AW_3 is given by

$$\{y_1, y_2, y_1^3, y_2^3, y_4^3 \mid y_1 y_2 y_1 y_2 y_1^{-1} y_2^{-1} y_1^{-1} = y_1^3,\ y_1 y_2 y_1 y_2^{-1} y_1^{-1} = y_2^3,\ y_1^3 y_2^3 (y_1^3)^{-1} = x_4^3,\ y_1^3 = y_2\}$$

(cf. fig. 5d). To obtain AW_3 we add to AW_2 three new generators y_1^3, y_2^3, x_4^3 and three relations $y_1^2 = y_2^3$, $x_3^2 = y_1^3$, $x_4^3 = x_4^2$. Using these relations we eliminate x_4^2, y_1^2 and x_3^2 giving the presentation

$$\{y_1, y_2, y_2^2, y_1^3, y_2^3, x_4^3 \mid y_1 y_2 y_1 y_2^{-1} y_1^{-1} = y_2^3,\ y_1 y_2 y_1^{-1} = y_2^2,\ y_2^3 y_2^2 (y_2^3)^{-1} = y_1^3,\ y_1^3 y_2^3 (y_1^3)^{-1} = x_4^3,\ y_1^3 = y_2\}.$$

Now substitution of the expression for y_2^2 into the third relation and elimination of y_2^2 (i.e. substitution of second relation in the third) gives the presentation AW_3. Finally elimination of y_1^3 in AW_3 (i.e. substitution of the last relation into the first one) and elimination of y_2^3 and x_4^3 (i.e. dropping second and third relations) gives Artin's presentation.

2. The presentation of the fundamental group of the complement to plane curves

Let C be an algebraic curve in $\mathbb{C}^2$ of degree d which we shall assume is in general position relative to the line at infinity in $\mathbb{CP}^2$.

Let π denote a linear projection $\mathbb{C}^2 \to \mathbb{C}^1 = \mathscr{L}$ and $p_1, \ldots, p_N$ be the points in $\mathscr{L}$ such that the fibres of π over p_i are either tangent to C or contain singular points of C. These tangency points or singular points of C will be called singularities of C relative to the chosen pencil. For simplicity we shall assume that π is chosen in such a way that at each tangency point the curve C and the line $\pi^{-1}(p_i)$ can be locally given by the equations $y = x^2$ and $y = 0$ respectively. Also we shall assume that $\pi^{-1}(p)$ contains at most one singular point of C and that $\pi^{-1}(p_i)$ does not belong to the tangent cone of C at this singular point. Let p_0 be any point in $\mathscr{L}$ distinct from $p_i (i = 1, \ldots, N)$ and $L_0 = \pi^{-1}(p_0)$. We shall fix a base point B in L_0 and consider a system of non-intersecting loops in $L_0 - L_0 \cap C$ representing a basis $x_1, \ldots, x_N$ of $\pi_1(L_0 - L_0 \cap C, B)$. The natural projection $\pi \colon \pi^{-1}\left(\mathscr{L} - \bigcup_{i=1}^{N} p_i\right) \to \mathscr{L} - \bigcup_{i=1}^{N} p_i$ defines a locally trivial bundle. Let D_i denote a small disk in L centered at p_i and let s_i be a system of non-intersecting paths connecting p_0 with a point q_i on the boundary of D_i (see fig. 6).

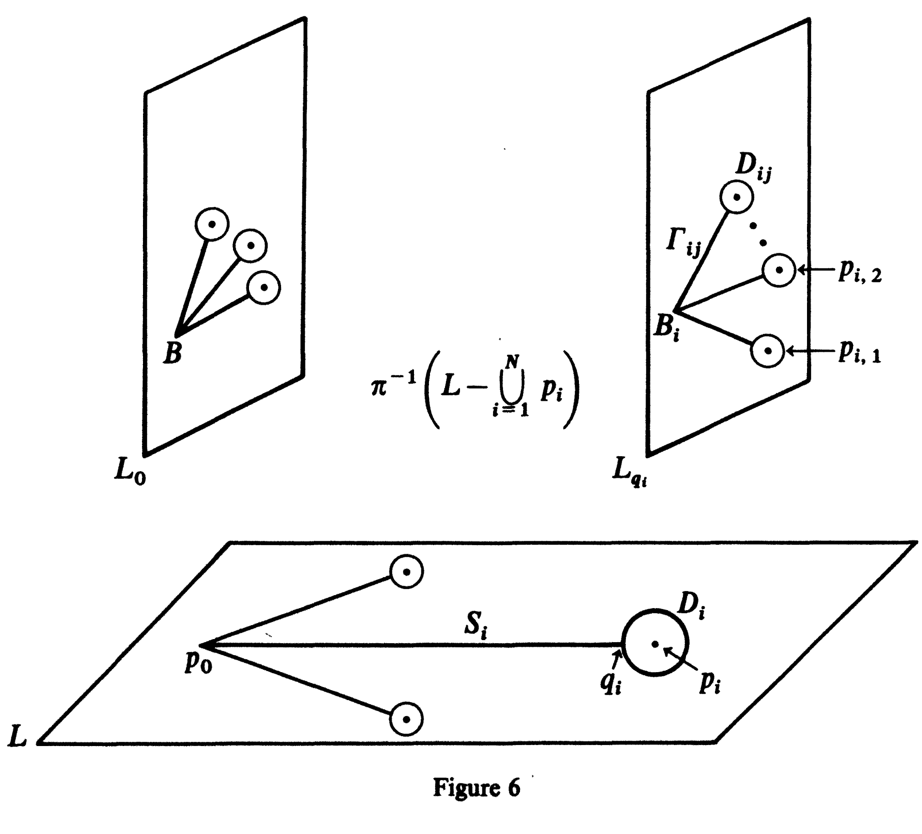

Figure 6

The intersection of the line $L_{q_i} = \pi^{-1}(q_i)$ and C consists of d points, m_i of which have as limit the singular point of C over p_i when q_i approaches p_i. Here m_i denotes the multiplicity of the corresponding singular point; for a tangency point $m_i = 2$. Let us label these m_i points in $\pi^{-1}(q_i)$ as $p_{i,1}, \ldots, p_{i,m_i}$. Let $\Gamma_{i,j}$, $j = 1, \ldots, d$ be a system of non-intersecting paths in L_{q_i} connecting a base point B_i (chosen, say, in vicinity of the base point of the pencil π) of L_{q_i} with the boundary of a small disk $D_{i,j}$ about points $L_{q_i} \cap C$ ($i = 1, \ldots, N$; $j = 1, \ldots, d$). These paths define the system of generators $y_{i,j} = \Gamma_{i,j} \cup \partial D_{i,j} \cup \Gamma_{i,j}^{-1}$ of $\pi_1(L_{q_i} - L_{q_i} \cap C, B_i)$. We may assume that the generators $y_{i,j}$ are labeled in such a way that $y_{i,1}, \ldots, y_{i,m_i}$ correspond to the points $p_{i,1}, \ldots p_{i,m_i}$.

Now we fix a system of non-intersecting segments in L_{q_i} (resp. in L_0) connecting points $L_{q_i} \cap C$ (resp. $L_0 \cap C$). These segments define a system of generators of the braid group $B(L_{q_i}, L_{q_i} \cap C)$ (resp. $B(L_0, L_0 \cap C)$) which is interpreted as the group of diffeomorphisms of L_{q_i} fixing $L_{q_i} \cap C$ (resp. $B(L_0, L_0 \cap C)$). Each segment defines a diffeomorphism which is a half twist about this segment. Let us fix also a diffeomorphism φ_i of $(L_0, L_0 \cap C)$ onto $(L_{q_i}, L_{q_i} \cap C)$ which takes the chosen generators of $B(L_0, L_0 \cap C)$ into the chosen generators of $B(L_{q_i}, L_{q_i} \cap C)$. A trivialization of the bundle $C^2 - C|_{\partial D_i - q_i}$ defines a diffeomorphism of $(L_{q_i}, L_{q_i} \cap C)$, i.e. a braid $\bar{\beta}_i \in B(L_{q_i}, L_{q_i} \cap C)$, which we shall call the local braid of the point p_i. Note that for appropriate choice of $\Gamma_{i,j}$ the braid $\bar{\beta}_i$ acts trivially on $y_{i,m_{i+1}}, \ldots, y_{i,d}$. In practice (cf. [M] and section 3) the aforementioned choice of generators of $B(L_{q_i}, L_{q_i} \cap C)$ is made in such a way that the local braid $\bar{\beta}_i$ will have the simplest possible form. For example if the singularity of C corresponding to p_i is given locally by $x^2 + y^s$ then the corresponding local braid is Y^s where Y is the half-twist about the segment connecting $p_{i,1}$ and $p_{i,2}$. For the points p_i corresponding to tangents to C the local braid is Y.

A trivialization of the locally trivial bundle $\mathbb{C}^2 - C \to \mathscr{L}$ restricted to s_i defines diffeomorphisms Φ_i: $(L_{q_i}, L_{q_i} \cap C) \to (L_0, L_0 \cap C)$. Let us consider the diffeomorphism $\Phi_i \bar{\beta}_i \Phi_i^{-1}$ of $(L_0, L_0 \cap C)$. We shall write it as $\Phi_i \varphi_i \varphi_i^{-1} \bar{\beta}_i \varphi_i (\Phi_i \varphi_i)^{-1}$ and put $Q_i = \Phi_i \varphi_i$ and $\beta_i = \varphi_i^{-1} \bar{\beta}_i \varphi_i$, $Q_i, \beta_i \in B(L_0, L_0 \cap C)$. Summarizing the situation, for any choice of paths s_i, a system of diffeomorphisms φ_i, Φ_i ($i = 1, \ldots, n$), and a compatiable system of segments in L_{q_i}, one constructs elements $Q_i \beta_i Q_i^{-1} \in B(L_0, L_0 \cap C)$ (cf. [M]). The homomorphism $\pi_1\left(\mathscr{L} - \bigcup_{i=1}^{N} p_i\right) \to B(L_0, L_0 \cap C)$ sending the i-th generator to $Q_i \beta_i Q_i^{-1}$ is called the braid monodromy (cf. [M]). Note that $\prod_{i=1}^{n} Q_i \beta_i Q_i^{-1} = \Delta^2$ where Δ^2 is the generator of the center of $B(L_0, L_0 \cap C)$ (cf. [M], [Ch]).

Now we are in position to formulate the main result.

Theorem. *The two-dimensional complex associated with the presentation of* $\pi_1(\mathbb{C}^2 - C)$ *with generators* $e_1, \ldots, e_d$ *and the relation*

$$Q_i(\beta_i e_j) = Q_i(e_j), \quad j = 1, \ldots, m_i - 1; \quad i = 1, \ldots, N$$

has the homotopy type of $\mathbb{C}^2 - C$.

Proof. We shall construct a series of retractions, the composition of which gives the retraction of $\mathbb{C}^2 - C$ onto the 2-complex defined in the statement of the theorem.

First we retract $\mathbb{C}^2 - C$ onto $\pi^{-1}\left(\bigcup_{i=1}^{N} s_i \cup D_i\right)$. The retraction is induced by the retraction of $L = \mathbb{C}$ onto $\bigcup_{i=1}^{N} s_i \cup D_i$. It retracts the locally trivial fibration $\pi^{-1}\left(\mathbb{C} - \bigcup_{i=1}^{N} (s_i \cup D_i)\right) \to \mathbb{C} - \bigcup_{i=1}^{N} (s_i \cup D_i)$ onto $\pi^{-1}\left(\bigcup_{i=1}^{N} s_i \cup \partial D_i\right)$ and leaves fixed

$$\pi^{-1}\left(\bigcup_{i=1}^{N} s_i \cup D_i\right).$$

Second, about the i-th singular point of C relative to the chosen pencil, we fix a small polydisk $\tilde{D}_i$ in $\mathbb{C}^2$ which projects onto D_i. Now we shall consider retractions of $\pi^{-1}(D_i)$ onto $\tilde{D}_i \cup L_{q_i}$ $(i = 1, \ldots, N)$. They exist provided $\tilde{D}_i$ is chosen so small that $\pi^{-1}(D_i) - \tilde{D}_i \to D_i$ is a trivial fibration.

Third, we make a retraction of $\tilde{D}_i - C$ onto $\partial \tilde{D}_i - C$ $(i = 1, \ldots, N)$. (Recall $C \cap \tilde{D}_i$ is a cone over $\partial \tilde{D}_i \cap C$.)

Fourth, it follows from lemma 2 that there is a retraction of each of $\partial \tilde{D}_i - C$ onto a 2-complex which has 1-cells in $\partial D_i \cap L_{q_i}$ and 2-cells e_{ij} attached according to the maps corresponding to the relations $\beta_i(e_{ij}) = e_{ij}$ $(j = 1, \ldots, (m_i - 1),\ i = 1, \ldots, N)$ where e_{ij} $(j = 1, \ldots, d,\ i = 1, \ldots, N)$ are certain generators of $\pi_1(L_{q_i} - C)$.

Finally using the trivialization of $\mathbb{C}^2 - C|_{s_i}$ from which the φ_i's were constructed we retract $\pi^{-1}(s_i) \cup e_{ij}$ onto $L_0 - C$ (with 2-cells attached). Then we retract $L_0 - C$ onto the union of cells represented by the generators e_j $(j = 1, \ldots, d)$. The former retraction takes e_{ij} to $Q_i(e)$ and $\beta_i(e_j)$ to $Q_i(\beta_i(e_j))$. Hence the two-complex obtained is the one described in the theorem. Q.E.D.

3. Change of homotopy type under degenerations and an example

For simplicity in this section we shall consider only degenerations in families of plane curves in which the only singularities of generic and special fibres are ordinary cusps or nodes (i.e. given respectively locally by $x^2 = y^3$ and $x^2 = y^2$).

Let π be a pencil of lines defining a projection $\pi\colon \mathbb{C}^2 \to \mathbb{C}$.

Proposition. *Let C_t be a family of curves which are in general position relative to the line in infinity for any t sufficiently close to zero. Assume that C_0 has either*

a) *a node obtained as a limit of two tangency points of the pencil π or*

b) *a cusp obtained as limit of a node of C_t and a tangency point of the pencil π.*

If $\pi_1(\mathbb{C}^2 - C_t)$ is isomorphic to $\pi_1(\mathbb{C}^2 - C_0)$ then $\mathbb{C}^2 - C_t$ is homotopy equivalent to

$$(\mathbb{C}^2 - C_0) \vee S^2.$$

Proof. Without loss of generality we can replace the disks D_i in the definition of braid monodromy by any small contractible regions and in particular we can assume that the point $q_i \in \partial D_i$ does not change while C_t is deformed into C_0. Therefore the regeneration (operation opposite to degeneration) $C_0 \to C_t$ changes the braid monodromy by replacing the factor $Qx_1^2Q^{-1}$ by the product of Qx_1Q^{-1} and Qx_1Q^{-1} in case a) (resp. replacing the factor $Qx_1^3Q^{-1}$ by the product of $Qx_1^2Q_1^{-1}$ and $Q_1x_1Q_1^{-1}$ in case b). This implies the following change in the presentation of $\pi_1(\mathbb{C}^2-C_t)$ constructed in section 2. The relation r_1: $Q(e_1)Q(e_2)=Q(e_2)Q(e_1)$ is replaced by two copies of the relation r_2: $Q(e_1)=Q(e_2)$. In the case b) the relation r_3:

$$Q(e_1)Q(e_2)Q(e_1)=Q(e_2)Q(e_1)Q(e_2)$$

is replaced by two relations r_1 and r_2.

Now we recall that any two finite presentations of a finitely generated group are related by a sequence of Tietze transformations (cf. [D]) of the following types:

(I) Adding (deleting) a generator and a relation which represents this generator in terms of other generators.

(II) Replacing a relation r by relation $rw^{-1}sw$ or $rw^{-1}s^{-1}w$ where s is another relation and w any word.

(III) Adding (deleting) the relation $1=1$.

The transformations of type (I) or (II) do not change the homotopy type of a 2-complex associated to a presentation of a group, while (III) amounts to the taking a wedge with S^2 (resp. splitting off S^2). Note that transformations (I) were used in the proof of lemma 2.

The relation r_1 can be obtained from relation r_2 using the following transformation of type (II):

$$\begin{aligned}Q(e_1)Q(e_2)^{-1} &\to Q(e_1)Q(e_2)^{-1}Q(e_2)(Q(e_1)Q(e_2)^{-1})^{-1}Q(e_2)^{-1}\\ &=Q(e_1)Q(e_2)Q(e_1)^{-1}Q(e_2)^{-1}.\end{aligned}$$

Now assume that we are in the case a). Then $\mathbb{C}^2-C_t$ has the homotopy type of a 2-complex corresponding the the presentation of $\pi_1(\mathbb{C}^2-C_t)$ with relations $R_1,\dots,R_l$ corresponding to singularities not affected by the degeneration under consideration and two identical relations r_2. On the other hand $\mathbb{C}^2-C_0$ has the homotopy type of a 2-complex corresponding to the presentation with relations R_{l_i} $i=1,\dots,l$ and one relation r_1. Now the Tietze transformation of type (I) transforms the set of relations $(R_1,\dots,R_l,r_2,r_2)$, into the set of relations $(R_1,\dots,R_l,r_1,r_2)$. Because $\pi_1(\mathbb{C}^2-C_t)$ is isomorphic to $\pi_1(\mathbb{C}^2-C_t)$, the relation r_2 belongs to the normal closure of $R_1,\dots,R_l,r_1$. It can be replaced using a transformation of type (II) by the relation $1=1$. The deleting of this relation produces the 2-complex homotopy equivalent to $\mathbb{C}^2-C_0$. The arguments in the case b) are completely similar.

Example. Let C be an affine portion of the branching curve of generic projection on $C\mathbb{P}^2$ of a non-singular cubic surface in $C\mathbb{P}^3$. The C is a curve of degree 6 with 6 cusps and no nodes. In [M] B. Moishezon computed the braid monodromy for this curve which gives the following presentation of the generator of the center:

$$(*)\qquad \Delta^2=[Z_{24}Z_{13}Z_{56}Z_{35}Z_{34}^3Z_{56}^3]^2\,Z_{24}Z_{13}Z_{50}Z_{35}$$

where Z_{ij} denote the braid which is the half-twist about the segment z_{ij} connecting points i and j or in terms of X_i, $i=1,\dots,d-1$,

$$Z_{ij}=X_{j-1}\cdots X_{i+1}X_i(X_{j-1}\cdots X_{i+1})^{-1}.$$

To write down the presentation of $\pi_1(\mathbb{C}^2 - C)$ for which the associated 2-complex has the homotopy type of $\mathbb{C}^2 - C$ we should rewrite each factor in the presentation (*) of Δ^2 in the form $Qx_1^\alpha Q^{-1}$. The relation corresponding to this factor is $Q(X_1^\alpha e_1) = Q(e_1)$. If Q_{ij} satisfies $Z_{ij} = Q_{ij} X_1 Q_{ij}^{-1}$ then $Q_{ij}(e_1) = e_i$ and $Q_{ij}(e_2) = e_j$. Hence the relation corresponding to Z_{ij} (resp. Z_{ij}^3) is $e_i = e_j$ (resp. $e_i e_j e_i = e_j e_i e_j$). Using this remark one obtains that $\mathbb{C}^2 - C$ is homotopy equivalent to the 2-complex given by generators $e_1, \ldots, e_6$ and relations $e_2 = e_4$, $e_1 = e_3$, $e_5 = e_6$, $e_3 = e_5$ (each counted 3 times) and $e_1 e_2 e_1 = e_2 e_1 e_2$, $e_5 e_4 e_3 = e_4 e_3 e_4$, $e_5 e_6 e_5 = e_6 e_6 e_6$ (each counted 2 times). The associated 2-complex is clearly homotopy equivalent to the 2-complex associated with the presentation $\{(e_1 e_2) | e_1 e_2 e_1 = e_2 e_1 e_2\}$ wedged with S^2 thirteen times. Hence

$$\mathbb{C}^2 - C = (S^3 - \{\text{trefoil knot}\}) \vee 13\, S^2.$$

We conclude with the following question. Do two curves in $\mathbb{C}^2$ exist whose complements have the same fundamental group and Euler characteristic but are not homotopy equivalent? Note that an example of two 2-complexes with fundamental group isomorphic to the group of trefoil knot and with Euler characteristic equal to 1 which are not homotopy equivalent was constructed by Dunwoody [D]. It is not clear however whether these two complexes can be realized as the complements to algebraic curves. The proposition above suggests a topological obstruction for degeneration of a curve without change of the fundamental group, namely an obstruction to decompose $\mathbb{C}^2 - C$ into a wedge of a two dimensional complex and an appropriate number of copies of S^2. Can this obstruction be non-trivial?

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