On topological complexity of solving polynomial equations of special type.

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Abstract. *The notion of Smale's topological comlplexity is reviewed. Topological and algebro-geometrical problems arising from finding topological complexity for solving polynomial equations with several vanishing coefficients formulated. Partial results toward their solutions are stated with an outline of proofs.*

In [**S**] S.Smale introduced the notion of topological complexity of an algorithm which provides an information on the *structure* of possible algorithms for solving a given problem rather then on their implementation time. Roughly speaking one assumes that the computation tree consists of nodes and connecting edges and that the nodes are either input nodes (having no incoming edges), or computation nodes (having one incoming and one outcoming edge), or branching nodes(having one incoming and two outcoming edges) or leaves (halts with no outcoming edges). The topological complexity of an algorithm is the number of branching nodes in its computation tree (or the number of leaves minus one).

In the same work S.Smale shows how the low bound for the topological complexity can be reduced to purely topological problems. For an algorithm for finding with accuracy ϵ the roots of a polynomial from a family of polynomials F one can state that the topological complexity is greater or equal than the Schwartz genus of the covering map which relates to an ordered collection of roots of a polynomial from F without multiple roots the collection of its coefficients. Here by the Schwartz genus of a map $f : X \to Y$ one means the minimal number k such that Y affords a cover with k open sets $U_1, ...U_k$, ($Y = \cup_{i=1}^{k} U_i$. such that f has a section over each U_i, i.e. for each i there exist a continuous map $g_i : U_i \to X$ such that $f \circ g_i = id$).

The Schwartz genus can be estimated from below as the maximal length of a non zero cup product of elements in $Ker(H^i(Y, Z_2)) \to H^i(X, Z_2))$. One can use here twisted coefficients instead Z_2 (cf.[**Sch**]). Using this method S. Smale ([**S**]) obtained $(log_2 n)^{2/3}$ as the lower bound for the topological complexity for finding with accuracy ε the roots of the polynomial equation with one unknown. On the other hand in the case when Y is a quotient of X by a free action of a discrete group G one can use the homological genus of any G-module A as a lower bound for the Schwartz genus of the quotient map. The A-homological genus of a principal G-bundle $f : X \to Y$ with the fibre a discrete group G with corresponding classifying map $c : Y \to K(G, 1)(K(G, 1)$ is the Eilenberg MacLane space of the group G) is the minimal integer i such that the canonical map $H^j(K(G, 1), A)) \to H^j(Y, c^*(A))$ is trivial for $j \geq i$ ([**Sch**]). Using this V.Vasiljev [**V**] obtained as a lower bound for the Smale's problem $n - min_p(D_{p(n)})$ where $D_{p(n)}$ is the sum of the digits in p-adic expansion of n and the minimum is taken over all primes p. He used as A the group of integers Z with the action of the symmetric group corresponding to the sign representation.

* Supported by NSF and U.S. Army grants.

It seems it would be interesting to estimate the topological complexity of the solving some special classes of polynomial equations, for example polynomial equations with several vanishing coefficients, or answer similar questions for systems of polynomial equations (the latter was addressed in [**L**]). The application of the Smale's theory requires rather detailed information on the topology of the complements to discriminants in the space of special types of polynomials which seems is not available at the moment. This is the problem which we begin to address here. Specifically the following should be answered.

Problem 1. What is the fundamental group of the space of polynomials with several vanishing coefficients? Do the cohomology of this space depend only on this fundamental group? i.e. is the space of polynomials with vanishing coefficients is the Eilenberg MacLane space.

Problem 2. What are the cohomology with various (twisted) coefficients of the space of polynomials with several vanishing coefficients? What is their relationship with the cohomology of symmetric group?

If one considers the space of all monic polynomials then the answer to problem 1 goes back to E.Artin ([**A**]) and Fadell and Neuwirth [**FN**]: the fundamental group of the space of monic polynomials without multiple roots is the braid group B_n on n strings and this space is the Eilenberg MacLane space of B_n. The cohomology of the symmetric group surjects on the cohomology of the braid group in the case of cohomology with Z_2 coefficients ([**S**]) or coefficients in sign representation of symmetric group ([**V**]).

Here we shall only indicate a solution for trinomials. First note that in the case of polynomials with several vanishing coefficients of the form

$$x^n + a_{i_1}x^{i_1} + a_{i_2}x^{i_2} \ldots + a_{i_k} \tag{1}$$

the discriminant hypersurface is rather different than the discriminant of the space of all monic polynomials of degree n: in may become reducible and have different than in generic case degree (when the degree is $2n-2$).

Examples of discriminants:

1) For

$$x^5 + ax^2 + bx + c$$

the discriminant is

$$-27a^4b^2 + 2250a^2bc^2 - 1600ab^3c + 3125c^4 + 256b^5 + 108ca^5$$

2) For

$$x^6 + \bar{a}x^3 + bx + c$$

the discriminant is

$$27000b^3ac^2 - 1350b^3ca^3 + 108a^5b^3 + 3125b^6 + 34992a^2c^4 - 87483 + 729c^2a^6 - 46656c^5$$

3) For

$$x^6 + ax^3 + bx^2 + c$$

the discriminant is

$$c^9 - 1024b^6 - 13824b^3c^2 + 108a^4b^3 - 46656c^4 + 729a^6c + 34992a^2c^3 - -8748a^4c^2 - 8640a^2b^3c$$

More generally one has the following:

Theorem A. *The discriminant of the family of polynomials of the form (1) has at most two irreducible components. The number of irreducible components is two if and only if $i_{k-1} \neq 1$ and in this case one of components is the linear subspace $a_{i_k} = 0$. The degree of the discriminant is $n + i_1 - i_{k-1}$.*

(The first part of this theorem is obtained in [FS]).In the case of trinomials

$$x^n + ax^k + b \tag{2}$$

one can give complete answer to the problem 1 above.

Theorem B. *The fundamental group of the space of polynomials of form (2) with no multiple roots is the group of an algebraic link of the type explicitly determined by n and k. In particular if $k = 1$ then the fundamental group of the space of polynomials of form (2) without multiple roots is the group of the torus knot of type $(n, n-1)$ i.e. admits a presentation with two generators g_1, g_2 and one relator $g_1^n = g_2^{n-1}$. This space is the Eilenberg MacLane space for any n and k.*

Remark: For $k = 1$ by virtue of having so simple presentation for the fundamental group one can easily describe the homomorphism of it into the braid group induced by embedding space of polynomials of form (2) into the space of all polynomials of degree n. If $s_1, ..., s_{n-1}$ are the standard generators of B_n then this homomorphism is given by $g_1 \to s_1 ... s_{n-1}, g_2 \to s_1 ... s_{n-1} s_1$. In particular this map is surjective. This in turn implies that the Galois group of generic trinomial equation in characteristic zero is the full symmetric group. (cf. [**Sm**] with much milder restrictions on characteristic of the ground field). This argument can be carried out in the case $k > 1$ as well.

Sketch of the proof First notice that the equation of the reduced discriminant of the polynomial (2) is

$$b((-1)^{n-k-1} k^k (n-k)^n a^n \quad + n^n b^{(n-k)}) = 0 \tag{3}$$

if $k > 1$ (cf. [S]). This follows from the fact that a polynomial has multiple root if and only if it and its derivative have common root. One can eliminate x from $x^n + ax^k + b = 0, nx^{n-1} + kax^{k-1} = 0$ by replacing last equation by $x^{n-k} = -ka/n$ (this is possible assuming $x \neq 0$ which is the case provided $b \neq 0$. $b = 0$ clearly belongs to support of discriminant if and only if $k > 1$ which accounts for the first factor in (3)), substituting this in the first equation and replacing it by expression for $x < k$ in terms of a and b after which elimination of x gives the second factor in (3). Now the complex curve D defined by (3) is invariant under $\mathbf{C}^*$ action on $\mathbf{C}^2$ which implies that the complement to D in $\mathbf{C}^2$ is equivalent to complement in 3-sphere to the link of the only singularity of the curve D namely the singularity at the origin. The Milnor fibration of the link of singularity of D exhibits the complement to the link of the singularity of D as a fibration over the circle with the real punctured surface as a fibre which implies that the complement to the curve D is the Eilenberg-MacLane space. In the case $k = 1$ the equation of the discriminant is given by the vanishing of the second factor in (3). This equation after change of variables looks like $u^n = v^{n-1}$. The link of singularity of this curve is the torus knot of type $(n, n-1)$ and the description of the fundamental group of the torus knot cited above is the well known one. The details of the proof of both theorems above and the cohomology calculations involved in the problem 2 will appear elsewhere.

Acknoledgment. The author thanks prof. S.Smale for introducing him to this subject and to Department of Mathematics of Columbia University for its hospitality which made it possible.

References.

[A] E. Artin, *Theory of Braids, Ann.of Math. 48, pp.101-126, (1947)*

[FN] *E.Fadell and L. Neuwirth, Configuration spaces, Math.Scand, 10 pp 111-118. (1962)*

[FS] *M.Freed and J.Smith, Irreducible discriminant components of the coefficient spaces, Acta Arithm. XLIV (1984) p.59-71.*

[L] *H.Levine, A Lower bound for the Topological complexity of Poly(D,n), Journal of Complexity 5, 34-44, (1989).*

[S] *S.Smale, On the topology of algorithms I, Journal of complexity, (3) (1987).*

[Sch] *A.S.Schwartz, Genus of fibre bundles, Proc. of Moscow Math. Soc. 10, 217-272 (1961).*

[Sm] *J.Smith, General trinomials having symmetric Galois group, Proc. Amer. Math. Soc. 63(1977) p.208-217. (1977).*